# Two Squares Canonical Factorization ${ }^{\star}$ 

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#### Abstract

We present a new combinatorial structure in a string: a canonical factorization for any two squares that occur at the same position and satisfy some size restrictions. We believe that this canonical factorization will have application to related problems such as the New Periodicity Lemma, Crochemore-Rytter Three Squares Lemma, and ultimately the maximum-number-of-runs conjecture.




## 1 Introduction

In 1995
I three distinct squares, all prefixes of a Lemma stating that, subject to certain of the other two. efixes of $\boldsymbol{x}$ with a a New Periodicity not exist. Since t to specify more hood of such two prese.

In this paper we of what we call doubl satisfying some size r torization can be trac double squares was pr we indicate how this
primitive strings ame position and f double squares and their unique fac1 of the factorization for more specific esent it in full generality. In conclusion the proof of New Periodicity Lemma.

2 Prelimin
In this section w a canonical facto tion Principle (s lead to the mair

A string $x$ infinite) set $\Sigma$, of $\boldsymbol{x}$, denoted $\mid \boldsymbol{x}$

binatorial tools that will be uare. Chief among these al Common Factor Lemma ( es Factorization Lemma (s mbols, called letters, drav e length of the sequence is nience we represent a strir an array $\boldsymbol{x}[1 . . n]$. The string of length zero is called the empty st a string $\boldsymbol{x}=\boldsymbol{u v} \boldsymbol{w}$, where $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are strings, then $\boldsymbol{u}$ (respectively, $\boldsymbol{v}, \boldsymbol{w}$ ) is said to

[^0]be a prefix (respectively, substring, suffix) of $\boldsymbol{x}$; a proper prefix (respectively, proper substring, proper suffix) if $|\boldsymbol{u}|<|\boldsymbol{x}|$ (respectively, $|\boldsymbol{v}|<|\boldsymbol{x}|,|\boldsymbol{w}|<|\boldsymbol{x}|)$. A substring is also called a factor. Given strings $\boldsymbol{u}$ and $\boldsymbol{v}, \operatorname{lcp}(\boldsymbol{u}, \boldsymbol{v})$ (respectively, $\operatorname{lcs}(\boldsymbol{u}, \boldsymbol{v})$ ) is the longest common prefix (respectively, longest common suffix) of $\boldsymbol{u}$ and $\boldsymbol{v}$.

If $\boldsymbol{x}$ is a concatenation of $k \geq 2$ copies of a nonempty string $\boldsymbol{u}$, we write $\boldsymbol{x}=\boldsymbol{u}^{k}$ and say that $\boldsymbol{x}$ is a repetition; if $k=2$, we say that $\boldsymbol{x}=\boldsymbol{u}^{2}$ is a square; if there exist no such integer $k$ and no such $\boldsymbol{u}$, we say that $\boldsymbol{x}$ is primitive. If $\boldsymbol{x}=\boldsymbol{v}^{2}$ has a proper prefix $\boldsymbol{u}^{2},|s u|<|\boldsymbol{v}|<2|\boldsymbol{u}|$, we say that $\boldsymbol{x}$ is a double square and write $\boldsymbol{x}=\mathrm{DS}(\boldsymbol{u}, \boldsymbol{v})$. A square $\boldsymbol{u}^{2}$ such that $\boldsymbol{u}$ has no square prefix is said to be regular.

For $\boldsymbol{x}=\boldsymbol{x}[1 . . n], 1 \leq i<j \leq j+k \leq n$, the string $\boldsymbol{x}[i+k . . j+k]$ is a right cyclic shift by $k$ positions of $\boldsymbol{x}[i . . j]$ if $\boldsymbol{x}[i]=\boldsymbol{x}[j+1], \ldots, \boldsymbol{x}[i+k-1]=\boldsymbol{x}[j+k]$. Equivalently, we can say that $\boldsymbol{x}[i . . j]$ is a left cyclic shift by $k$ positions of $\boldsymbol{x}[i+k . . j+k]$. When it is clear from the context, we may leave out the number of positions and just speak of a cyclic shift.

Strings $\boldsymbol{u v}$ and $\boldsymbol{v u}$ are conjugates, written $\boldsymbol{u v} \sim \boldsymbol{v} \boldsymbol{u}$. We also say that $\boldsymbol{v} \boldsymbol{u}$ is the $|\boldsymbol{u}|^{\text {th }}$ rotation of $\boldsymbol{x}$, written $R_{\mid} \boldsymbol{u}(\boldsymbol{x})$, or the $-|\boldsymbol{v}|^{\text {th }}$ rotation of $\boldsymbol{x}$, written $R_{-|\boldsymbol{v}|}(\boldsymbol{x})$, while $R_{0}(\boldsymbol{x})=R_{-|\boldsymbol{x}|}(\boldsymbol{x})=\boldsymbol{x}$ is a primitive rotation. Similarly as for the cyclic shift, when it is clear from the context, we may leave out the number of rotations and just speak of a rotation. Note that all cyclic shifts are conjugates, but not the other


Proof. First consider the special case $\boldsymbol{x}_{\mathbf{1}}=\boldsymbol{x}_{\mathbf{2}}=\boldsymbol{y}_{\mathbf{1}}=\boldsymbol{y}_{\mathbf{2}}=\boldsymbol{\varepsilon}$, where $\boldsymbol{x}^{p}, \boldsymbol{y}^{q}$ have a common prefix $\boldsymbol{f}$ of length $|\boldsymbol{x}|+|\boldsymbol{y}|$. We show that in this case $\boldsymbol{x}=\boldsymbol{y}$.

Observe that $\boldsymbol{f}$ has prefixes $\boldsymbol{x}$ and $\boldsymbol{y}$, so that if $|\boldsymbol{x}|=|\boldsymbol{y}|$, then $\boldsymbol{x}=\boldsymbol{y}$, as required. Therefore suppose WLOG that $|\boldsymbol{x}|<|\boldsymbol{y}|$. Note that $\boldsymbol{y} \neq \boldsymbol{x}^{k}$ for any integer $k \geq 2$, since otherwise $\boldsymbol{y}$ would not be primitive, contradicting the hypothesis of the lemma.
that $k|\boldsymbol{x}|<|\boldsymbol{y}|$ and $(k+1)|\boldsymbol{x}|>|\boldsymbol{y}|$. But since $\boldsymbol{f}=\boldsymbol{y} \boldsymbol{x}$,

$$
R_{|\boldsymbol{y}|-k|\boldsymbol{x}|}(\boldsymbol{x})=\boldsymbol{x},
$$


o the assumption that $\boldsymbol{x}$ is primitive. We conclude that and $\boldsymbol{x}=\boldsymbol{y}$, as required. case, where $\boldsymbol{f}$ of length $|\boldsymbol{x}|+|\boldsymbol{y}|$ is a common factor of ${ }_{2} \boldsymbol{x}^{p} \boldsymbol{x}_{\mathbf{1}}=\boldsymbol{u} \boldsymbol{f} \boldsymbol{u}^{\prime}$ for some $\boldsymbol{u}$ and $\boldsymbol{u}^{\prime}$. If $|\boldsymbol{u}| \geq|\boldsymbol{x}|$, then hd so we can assume WLOG that $|\boldsymbol{u}|<|\boldsymbol{x}|$. Setting $\tilde{\boldsymbol{x}}=R_{|\boldsymbol{u}|}(\boldsymbol{x})$ wo seo that $\boldsymbol{f}$ ic mrefix of $\tilde{\boldsymbol{x}}^{p}$.

Similarl
$f$ is also a above, for

Note th
Lemma 4
prefix and a integers $p$
$\boldsymbol{v} \boldsymbol{f} \boldsymbol{v}^{\prime}$, we can assume that $|\boldsymbol{v}|<|\boldsymbol{y}|$, hence that $\boldsymbol{v} \mid(\boldsymbol{y})$. But this is just the special case considered $\tilde{\boldsymbol{x}}$ and $\boldsymbol{y} \sim \tilde{\boldsymbol{y}}$, the result follows.
quivalently stated in a more general form:
the primitive root $\overline{\boldsymbol{x}}$ of $\boldsymbol{x}$ and the primitive root $\overline{\boldsymbol{y}}$ of $\boldsymbol{y}$ are conjugates.
The Common Factor Lemma gives rise to the following useful corollary:
Lemma 5 Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are primitive strings, and that $p$ and $q$ are positive integers.
(a) If $\boldsymbol{x}^{p}=\boldsymbol{y}^{q}$, then $\boldsymbol{x}=\boldsymbol{y}$ and $p=q$.
(b) If $\boldsymbol{x}_{\mathbf{1}}$ (respectively, $\boldsymbol{y}_{\mathbf{1}}$ ) is a proper prefix of $\boldsymbol{x}$ (respectively, $\boldsymbol{y}$ ) and $\boldsymbol{x}^{p} \boldsymbol{x}_{\mathbf{1}}=\boldsymbol{y}^{q} \boldsymbol{y}_{\mathbf{1}}$ for $p \geq 2, q \geq 2$, then $\boldsymbol{x}=\boldsymbol{y}, \boldsymbol{x}_{\mathbf{1}}=\boldsymbol{y}_{\mathbf{1}}$ and $p=q$.


## 3 Main Result - Iwo Squares Factorization

The next lemma specifies the structure imposed by the occurrei the same position in a string. This structure has been described but not as precisely and with more assumptions required; above lishes the uniqueness of the breakdown.

Lemma 6 (Two Squares Factorization Lemma) For a dor there exists a unique primitive string $\boldsymbol{u}_{\mathbf{1}}$ such that $\boldsymbol{u}=\boldsymbol{u}_{1}{ }^{e_{1}} \boldsymbol{u}_{\mathbf{2}}$ ar
 where $\boldsymbol{u}_{\mathbf{2}}$ is a possibly empty proper prefix of $\boldsymbol{u}_{\mathbf{1}}$ and $e_{1}, e_{2}$ are integers such that $e_{1} \geq e_{2} \geq 1$. Moreover,
(a) if $\left|\boldsymbol{u}_{\mathbf{2}}\right|=0$, then $e_{1}>e_{2} \geq 1$;
(b) if $\left|\boldsymbol{u}_{\mathbf{2}}\right|>0$, then $\boldsymbol{v}$ is primitive, and if in addition $e_{1} \geq 2$, then $\boldsymbol{u}$ also is primitive.

In both cases, the factorization is unique.
Proof. If we have $\boldsymbol{u}^{k}, k \geq 2$, we refer to the first copy of $\boldsymbol{u}$ as $\boldsymbol{u}_{[1]}$, to the second copy of $\boldsymbol{u}$ as $\boldsymbol{u}_{[2]}$ etc.

Let $\boldsymbol{z}$ be the nonempty proper prefix of $\boldsymbol{u}_{[2]}$ that is in addition a suffix $\boldsymbol{z}$ of $\boldsymbol{v}_{[1]}$. But then $\boldsymbol{z}$ is also a prefix of $\boldsymbol{v}_{[1]}$, hence of $\boldsymbol{v}_{[2]}$; thus if $|\boldsymbol{u}| \geq 2|\boldsymbol{z}|$, it follows that $\boldsymbol{z}^{2}$ is a prefix of $\boldsymbol{u}$. In general, there exists an integer $k=\lfloor|\boldsymbol{u}| /|\boldsymbol{z}|\rfloor \geq 1$ such that $\boldsymbol{u}=\boldsymbol{z}^{k} \boldsymbol{z}^{\prime}$ for some proper suffix $\boldsymbol{z}^{\prime}$ of $\boldsymbol{z}$. Let $\boldsymbol{u}_{\boldsymbol{1}}$ be the primitive root of $\boldsymbol{z}$, so that $\boldsymbol{z}=\boldsymbol{u}_{1}{ }^{e_{2}}$ for some integer $e_{2} \geq 1$. Therefore, for some $e_{1} \geq e_{2} k$ and some prefix $\boldsymbol{u}_{\mathbf{2}}$ of $\boldsymbol{u}_{\mathbf{1}}, \boldsymbol{u}=\boldsymbol{u}_{1}{ }^{e_{1}} \boldsymbol{u}_{\boldsymbol{2}}$ and $\boldsymbol{v}=\boldsymbol{u} \boldsymbol{z}=\boldsymbol{u}_{1}{ }^{e_{1}} \boldsymbol{u}_{\boldsymbol{2}} \boldsymbol{u}_{\boldsymbol{1}}{ }^{e_{2}}$, as required. To prove consider two cases:
(i) $\left|\boldsymbol{u}_{\mathbf{2}}\right|=0$

Here $\boldsymbol{u}=\boldsymbol{u}_{1}{ }^{e_{1}}$ and $\boldsymbol{v}=\boldsymbol{u}_{1}{ }^{e_{1}+e_{2}}$, so that $\boldsymbol{x}=\boldsymbol{u}_{1}{ }^{2\left(e_{1}+e_{2}\right)}$. Since $|\boldsymbol{v}|$ $e_{2}$, it follows that $e_{1}>e_{2}$. The uniqueness of $\boldsymbol{u}_{1}$ is a consequence
(ii) $\left|\boldsymbol{u}_{2}\right|>0$

Suppose the choice of $\boldsymbol{u}_{1}$ is not unique. Then there exists some pri with proper prefix $\boldsymbol{w}_{\mathbf{2}}$, together with integers $f_{1} \geq f_{2} \geq 1$, such tl and $\boldsymbol{v}=\boldsymbol{w}_{\mathbf{1}}{ }^{f_{1}} \boldsymbol{w}_{\mathbf{2}} \boldsymbol{w}_{\mathbf{1}}{ }^{f_{2}}$. If both $e_{1} \geq 2$ and $f_{1} \geq 2$, it follows frc that $\boldsymbol{u}_{\boldsymbol{1}}=\boldsymbol{w}_{\mathbf{1}}$ and $e_{1}=f_{1}$. If $e_{1}=f_{1}=1$, we observe that $\boldsymbol{v}$ so that again $\boldsymbol{u}_{\boldsymbol{1}}=\boldsymbol{w}_{\boldsymbol{1}}$. In the only remaining case, exactly one 1: therefore suppose WLOG that $f_{1}>e_{1}=1$. Then $\boldsymbol{u}=\boldsymbol{u}_{\boldsymbol{1}} \boldsymbol{u}_{\mathbf{2}}$ $\boldsymbol{v}=\boldsymbol{u}_{\mathbf{1}} \boldsymbol{u}_{\mathbf{2}} \boldsymbol{u}_{\mathbf{1}}=\boldsymbol{w}_{\mathbf{1}}{ }^{f_{1}} \boldsymbol{w}_{\mathbf{2}} \boldsymbol{w}_{\mathbf{1}}{ }^{f_{2}}$, so that $\boldsymbol{u}_{\boldsymbol{1}}=\boldsymbol{w}_{\mathbf{1}}^{f_{2}}$. But since $\boldsymbol{u}_{\boldsymbol{1}}$ is forces $f_{2}=1$ and $\boldsymbol{u}_{\mathbf{1}}=\boldsymbol{w}_{\mathbf{1}}$, which, since $\boldsymbol{u}_{\mathbf{1}} \boldsymbol{u}_{\mathbf{2}}=\boldsymbol{w}_{\mathbf{1}}{ }^{f_{1}} \boldsymbol{w}_{\mathbf{2}}=\boldsymbol{u}_{\mathbf{1}}{ }^{f_{1}} \boldsymbol{w}_{\mathbf{2}}$, implies that $f_{1}=1$, a contradiction. Thus all cases have been considered, and $\boldsymbol{u}_{\mathbf{1}}$ is unique.
We now show that $\boldsymbol{v}$ is primitive. Suppose the contrary, so there exists some primitive $\boldsymbol{w}$ and an integer $k \geq 2$ such that $\boldsymbol{v}=\boldsymbol{w}^{k}$. It follows that $|\boldsymbol{w}| \leq|\boldsymbol{v}| / 2 \leq\left|\boldsymbol{u}_{1}{ }^{e_{1}}\right|+\left|\boldsymbol{u}_{\mathbf{2}}\right|$. Note that


The following ex
(a) The second part not necessary, co
 tatement of the lemma is sharp:
s that $e_{1} \geq 2$. To see that this condition is , where $\boldsymbol{u}=(a b) a, \boldsymbol{v}=(a b) a(a b)$, so that orimitive.
baabaababaabaabaab, where $\boldsymbol{u}=(a b a)^{2}=$ $\boldsymbol{u}_{\mathbf{1}}=a b a a b, \boldsymbol{u}_{\mathbf{2}}=a, e_{1}=e_{2}=1$, where
ive.
ence to the
uble square
caranteed
 erminology and notation: unique factorization $\boldsymbol{v}^{2}=$ onical factorization of he symbol $\overline{\boldsymbol{u}}_{\mathbf{2}}$ denotes the rvations:
$f$ any one of the following conditions holds:
(c) there is no farther to the right in $\boldsymbol{v}^{2}\left(\boldsymbol{u}^{2}\right.$ is rightmost);
(d) $\boldsymbol{u}^{2}$ is regula

## Moreover:

(e) $\left|\boldsymbol{u}_{\boldsymbol{2}}\right|>0$ if and only if $\boldsymbol{v}$ is primitive;
(f) If $\boldsymbol{u}^{2}$ is regular, then $e_{1}=e_{2}=1$ and $\boldsymbol{u}_{\boldsymbol{1}}$ is regular.

Proof.
(a) $\left|\boldsymbol{u}_{\boldsymbol{2}}\right|=0$ implies $\boldsymbol{v}$ not primitive.
(b) $\left|\boldsymbol{u}_{\boldsymbol{2}}\right|=0$ implies $\boldsymbol{u}$ not primitive.
(c) $\left|\boldsymbol{u}_{\boldsymbol{2}}\right|=0$ implies $\boldsymbol{u}^{2}=\boldsymbol{u}_{1}{ }^{2 e_{1}}$, which occ as a suffix.
(d) Since $\boldsymbol{u}^{2}$ is regular, therefore $\boldsymbol{u}$ is primi
(e) By (a), primiti primitive.
(f) $\mathrm{By}(\mathrm{d})$, regular $e_{1}=e_{2}=1$ and
In the co aabaa, v observe t general t

Now,
Thus $\boldsymbol{v}^{2}$


implies that $\boldsymbol{v}$ is
is regular only if
$=1$, but
1 is more

$$
{ }_{1}=u_{2} \bar{u}_{2}
$$

$$
\begin{aligned}
& \left.{ }_{2} \overline{\boldsymbol{u}}_{\boldsymbol{2}}\right)^{e_{1}} \boldsymbol{u}_{\mathbf{2}}\left(\boldsymbol{u}_{\mathbf{2}} \overline{\boldsymbol{u}}_{\boldsymbol{2}}\right)^{e_{1}+} \\
& { }^{-1} \boldsymbol{u}_{\mathbf{2}}(\mathrm{IF})\left(\boldsymbol{u}_{\mathbf{2}} \overline{\boldsymbol{u}}_{\mathbf{2}}\right)^{e_{1}+\boldsymbol{u}}
\end{aligned}
$$

$$
\begin{equation*}
{ }^{-1} \boldsymbol{u}_{\mathbf{2}}(\mathrm{IF})\left(\boldsymbol{u}_{2} \overline{\boldsymbol{u}}_{\boldsymbol{2}}\right)^{e_{1}+e} \tag{2}
\end{equation*}
$$

where $\mathrm{IF}=$
$u_{1}$ is called the inversion factor.
Lemma 9
Then the in $|\boldsymbol{v}|$ apart as
$\operatorname{DS}(\boldsymbol{u}, \boldsymbol{v})=\left(\boldsymbol{u}_{\mathbf{1}}, \boldsymbol{u}_{\mathbf{2}}, e_{1}, e_{2}\right)$ with a non-empty $\boldsymbol{u}_{\mathbf{2}}$. xactly two occurrences in $\boldsymbol{v}^{2}$ exactly a distance of

Proof. If IF occurs elsewhere in $\boldsymbol{v}^{2}$, by the Synchronization principle $\boldsymbol{u}_{2} \overline{\boldsymbol{u}}_{\boldsymbol{2}}$ must align with an occurrence of $\boldsymbol{u}_{2} \overline{\boldsymbol{u}}_{\boldsymbol{2}}$ as it is primitive. Thus, $\overline{\boldsymbol{u}}_{2} \boldsymbol{u}_{\mathbf{2}}$ must align with $\boldsymbol{u}_{2} \overline{\boldsymbol{u}}_{\mathbf{2}}$, contradicting the primitiveness of $\boldsymbol{u}_{\mathbf{2}} \overline{\boldsymbol{u}}_{\mathbf{2}}$, s


## 4 Possible application to New Per

Some years ago a New Periodicity Lemma was currence of two special squares at a position $i$ ir $\begin{array}{ll}\text { occ } & \text { S of specific period in } \\ \text { pro } & \text { omplex, breaking dow }\end{array}$ the shorter of the two
) Let $\boldsymbol{x}=\mathrm{DS}(\boldsymbol{u}, \boldsymbol{v})$, where we require hen for all integers $k$ and $w$ such that $|\boldsymbol{u}|, \boldsymbol{x}[k+1 . . k+2 w]$ is not a square.
ement that $v$ be primitive is redundant; e primitivness of $v$. Also note that the canonical factorization of $\operatorname{DS}(\boldsymbol{u}, \boldsymbol{v})=$ regularity of $u^{2}$ necessa $\left(\boldsymbol{u}_{\mathbf{1}}, \boldsymbol{u}_{\mathbf{2}}, e_{1}, e_{2}\right), e_{1}=e_{2}$

Consider $\operatorname{DS}(\boldsymbol{u}, \boldsymbol{v})=\left(\boldsymbol{u}_{\boldsymbol{1}}, \boldsymbol{u}_{\mathbf{2}}, 1,1\right)$. Let $\boldsymbol{u}_{\boldsymbol{2}}$ be a suffix of $\boldsymbol{u}_{\boldsymbol{1}}$ such that $\boldsymbol{u}_{\boldsymbol{1}}=\boldsymbol{u}_{\mathbf{2}} \overline{\boldsymbol{u}}_{\boldsymbol{2}}$. The canonical factorization thus has the form

$$
\left(u_{2} \bar{u}_{2}\right) u_{2}\left(u_{2} \bar{u}_{2}\right)\left(u_{2} \bar{u}_{2}\right) u_{2}\left(u_{2} \bar{u}_{2}\right) .
$$

Let us consider a square $\boldsymbol{w}^{2}$ such that $\left|\boldsymbol{u}_{\mathbf{1}}\right|<|\boldsymbol{w}|<|\boldsymbol{v}|$ and $|\boldsymbol{w}| \neq|\boldsymbol{u}|$. We want to show that this is not possible.
If for instance $\boldsymbol{w}$ starts in the first $\boldsymbol{u}_{\mathbf{2}}$ and ends in the fourth $\boldsymbol{u}_{\mathbf{2}}$, then $\boldsymbol{w}$ contains fully the IF, so the second $\boldsymbol{w}$ has to as well, and so $|\boldsymbol{w}| \geq|\boldsymbol{v}|$, a contradiction.
If $\boldsymbol{w}$ ends in the second $\overline{\boldsymbol{u}}_{\boldsymbol{2}}$ we cannot argue using IF, but still knowing that $\boldsymbol{u}_{\mathbf{2}} \overline{\boldsymbol{u}}_{\boldsymbol{2}}$ is
primitive and also all its rotations are primitive, using the Synchronization principle can be applied to obtain a contradiction.

Almost all possible cases for $\boldsymbol{w}^{2}$ except two can be easily shown impossible using only the properties of the canonical factorization. Thus, we believe, and it is our immediate goal for future research, that the canonical factorization will not only provide us with a significantly simplified proof of New Periodicity Lemma, but will also allow us to significantly reduce the conditions on $\boldsymbol{u}^{2}$ from $\boldsymbol{u}$ being regular to just being primitive. We also believe that the canonical factorization in the same way will not only provide a simpler proof of Crochemore-Rytter Three Squares Lemma, but will extend the applicability of the lemma to three squares when any of the squares is primitive (the original lemma requires that the smallest square be primitive).

## 5 Conclusion and future work

We presented a unique factorization of a double square, i.e. a configuration of two squares $\boldsymbol{u}^{2}$ and $\boldsymbol{v}^{2}$ starting at satisfying $|u|<|v|<2|u|$. We
 has very strong combinatorial primitive string. We indicated structure of double squares in hemore-Rytter's Three Squares paring this final version of the e are happy to report that the simplified and generalized both. results in a near future.

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