Reordering a tree according to an order on its leaves

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Introduction

Initial Motivation

Linguistic Question:
How does an author style evolve through time?

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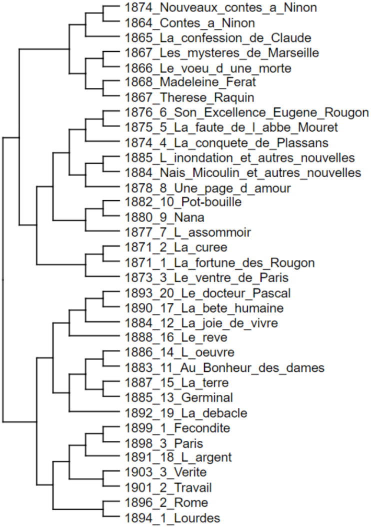
How does an author style evolve through time?

- novels are clustered by **linguistic criteria** (word and phrase frequencies, etc.)

→ *dendrogram*



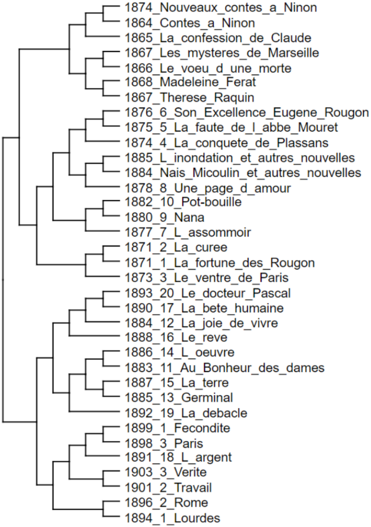
Initial Motivation

Linguistic Question:
How does an author style evolve through time?

- novels are clustered by **linguistic criteria** (word and phrase frequencies, etc.)

$\rightarrow$ *dendrogram*

- does the clustering group together novels published in **consecutive years**?



Modelization

Motivation

Is a clustering consistent with an external ordering?

Input:

- Elements

D

A E

B F

G

C

H

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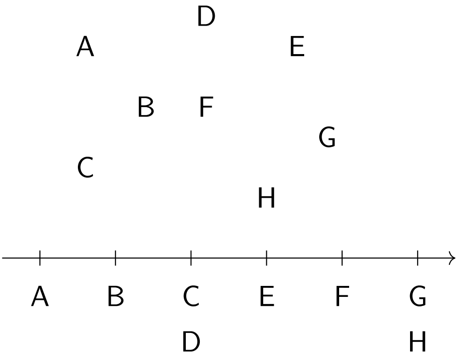
- Ordering (time-line, ...)



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Is a clustering consistent with an external ordering?

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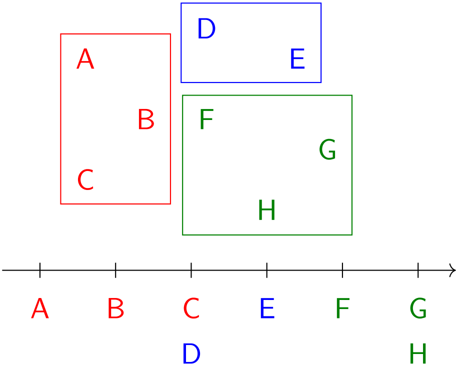

Is the clustering consistent with the ordering?

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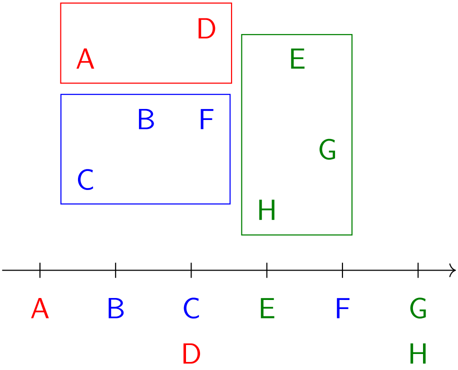

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Motivation

Is a clustering consistent with an external ordering?

Input:

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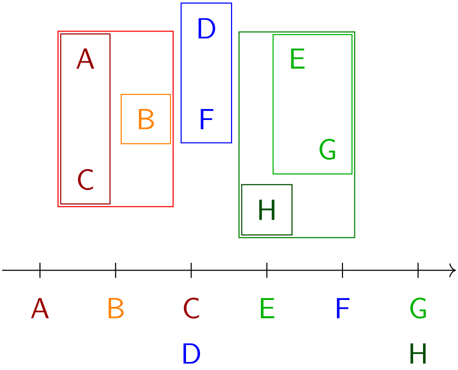
- Ordering (time-line, ...)
- Hierarchical Clustering



Is the clustering consistent with the ordering?

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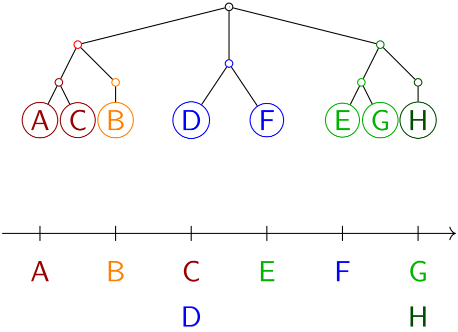
Is a clustering consistent with an external ordering?

Input:

- Elements
- Ordering (time-line, ...)
- Hierarchical Clustering (seen as a tree / dendrogram)



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Is a clustering consistent with an external ordering?

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- Non-strict Ordering (time-line, ...)
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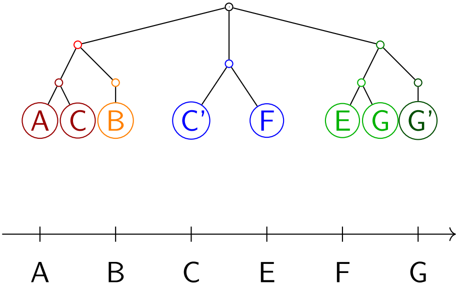

Definitions

- Tree T with leaf set X, ordering $\sigma \colon X \to \mathbb{N}$ (weak order $\leq_\sigma$)

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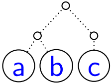
- Conflict: leaves a, b, c with $a <_\sigma c <_\sigma b$ and 

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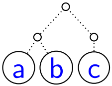
- Conflict: leaves a, b, c with $a <_\sigma c <_\sigma b$ and 

OTDE One-Tree Drawing by Deleting Edges

Given T, σ, k,
Find $X' \subseteq X$, $|X'| \geq |X| - k$
Such that $T[X']$ has no conflict with σ

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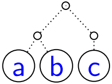
- Conflict: leaves a, b, c with $a <_\sigma c <_\sigma b$ and 

- Ordering of T: strict order σ' without conflict with T
  $\Leftrightarrow$ permute the children of each node, read leaves from left to right

TTDE Two-Tree Drawing by Deleting Edges

Given T_1, T_2, k,
Find $X' \subseteq X$, $|X'| \geq |X| - k$,
and an ordering σ of both $T_1[X']$ and $T_2[X']$

Definitions

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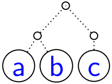
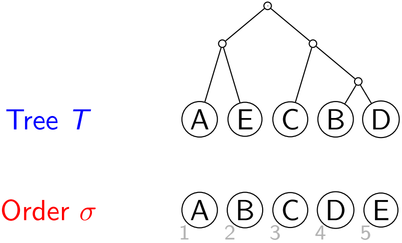
- Conflict: leaves a, b, c with $a <_\sigma c <_\sigma b$ and 

- Ordering of T: strict order σ' without conflict with T
  $\Leftrightarrow$ permute the children of each node, read leaves from left to right
- Crossing between σ and σ': pair $\{a, b\}$ with $a <_\sigma b$ and $b <_{\sigma'} a$

OTCM One-Tree Crossing Minimization

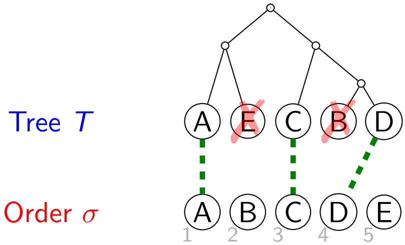
Given T, σ, k,
Find σ' ordering of T
Such that σ' has at most k crossings with σ

Example



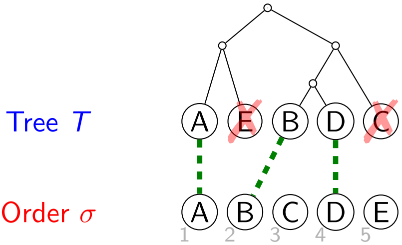
Input instance

Example



Score for OTDE: $k = 2$ deletions

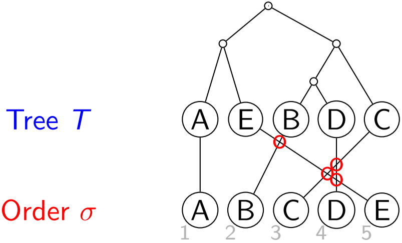
Example



Another solution with the same score

fun fact: all possible permutations of each node's children need 2 deletions

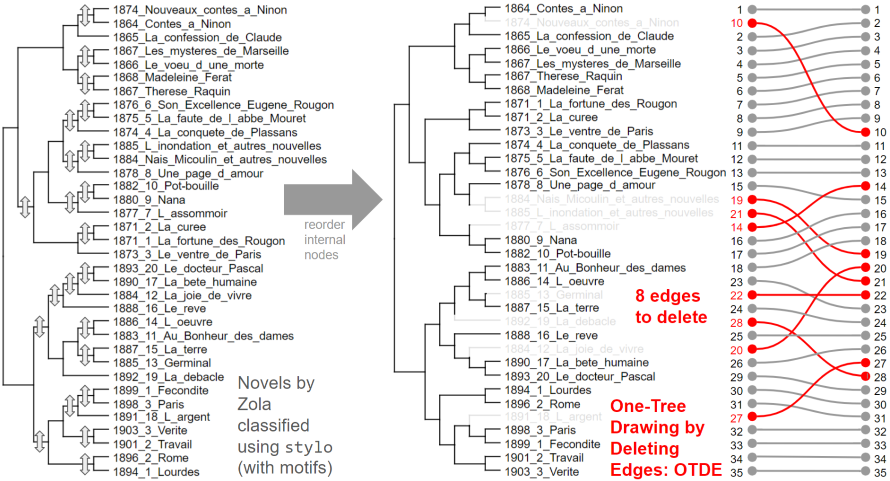
Example



Score for OTCM: 4 crossings

Initial Question

How much is the dendrogram consistent with time?



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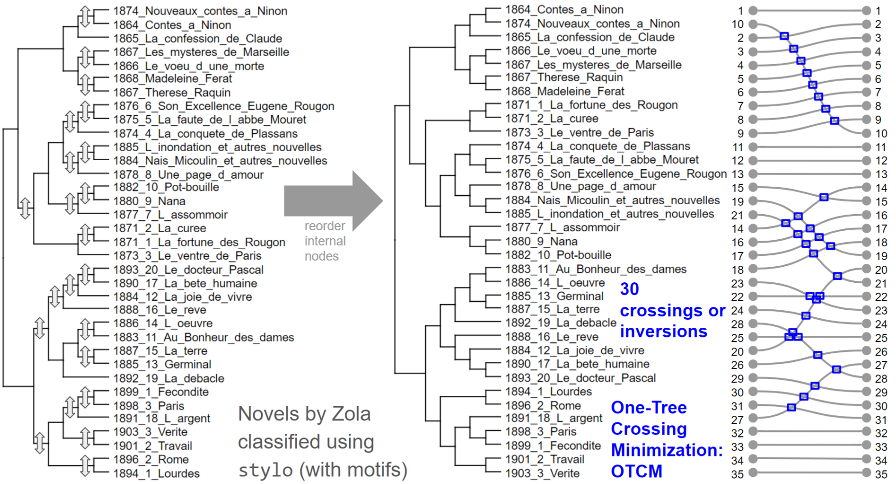

Previous Results

OTCM on binary trees

Most studied variant, from phylogenetics

- Dwyer, Schreiber '04: $O(n^2)$
- Fernau, Kaufmann, Poths '05: $O(n \log^2 n)$
- Bansal et al. '09: $O(n \log^2 n / \log \log n)$
- Fernau, Kaufmann, Poths. '10 and Venkatachalam, et al. '10: $O(n \log n)$

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OTDE, TTDE

Introduced by Fernau et al.:

- Reduction **from** OTDE **to** 3-Hitting Set
- NP-hardness still open

Our Results

With arbitrary-degree trees

OTCM

- NP-hardness (from Feedback Arc Set)

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With arbitrary-degree trees

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OTDE

- NP-hardness (from Independent Set)
- Parameterized algorithms
  - (simple) XP for the degree d [1]
  - (advanced) FPT for the *deletion-degree* ∂ [2]

[1] $O(d!n^{d+2})$
[2] $O(d^2 2^{\partial} n^4)$ with $\partial =$ degree of $T[X \setminus X']$, $\partial \leq \min\{d, k\}$

Our Results

With arbitrary-degree trees

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TTDE

- NP-hardness (from OTDE)

---

[1] $O(d! n^{d+2})$

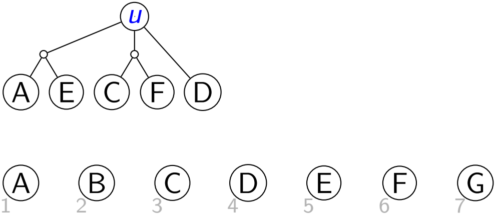
[2] $O(d^2 2^{\partial} n^4)$ with $\partial =$ degree of $T[X \setminus X']$, $\partial \leq \min\{d, k\}$

Algorithms

OTDE is XP for the degree

Bottom-up Dynamic Programming

For each internal node v, interval l, r
$X(v, l, r)$ = deletions in $T[v]$ when mapped with $\sigma[l..r]$



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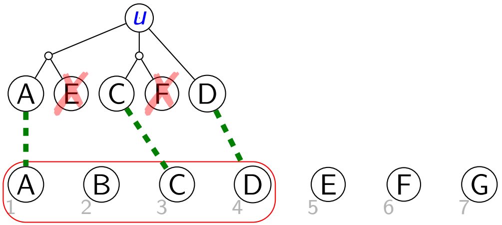

$X(u, 1, 4) = 2$

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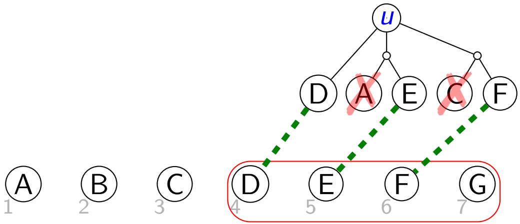

$X(u, 1, 4) = 2$
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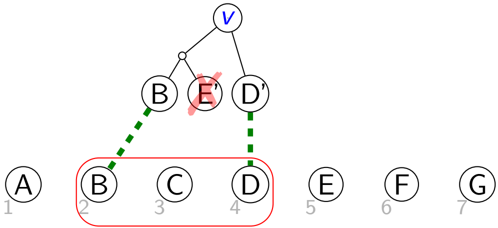

$X(u, 1, 4) = 2$ $\quad$ $X(v, 2, 4) = 1$

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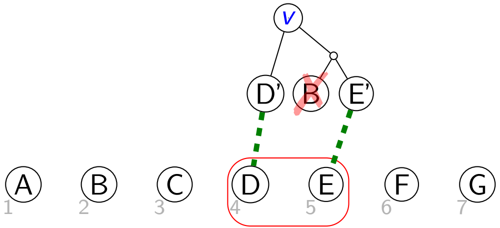

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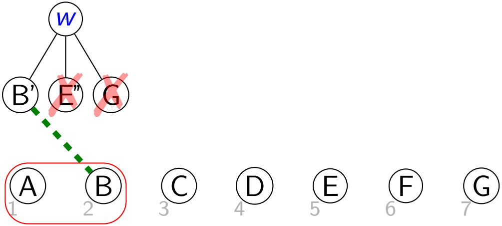

$X(u, 1, 4) = 2$ $\quad$ $X(v, 2, 4) = 1$ $\quad$ $X(w, 1, 2) = 2$
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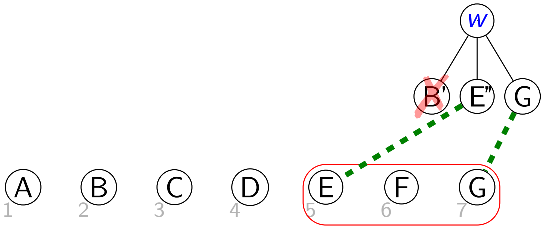

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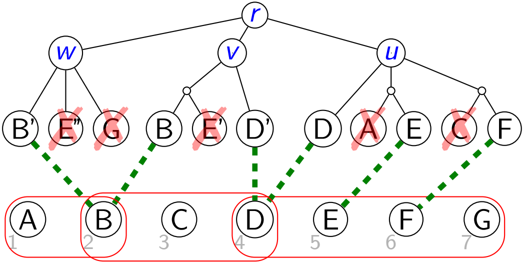

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$X(r, 1, 7) = \min(5, \ldots)$

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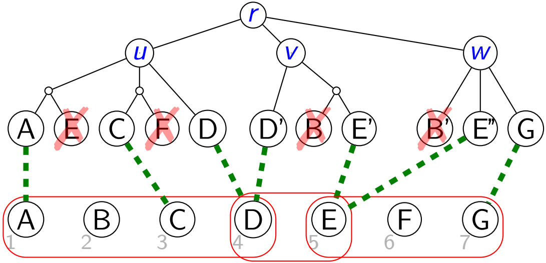

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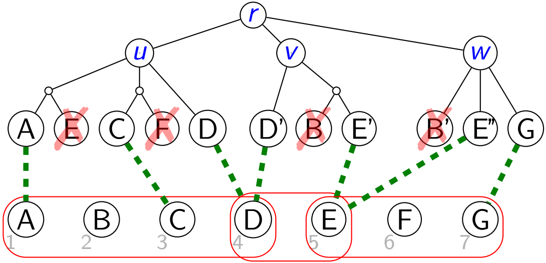

n^3 DP entries

$\boxed{X(u, 1, 4) = 2}$ $X(v, 2, 4) = 1$ $X(w, 1, 2) = 2$

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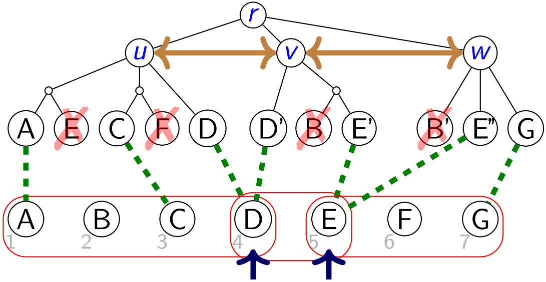

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$d!$ permutations of the children

n^{d-1} pivots

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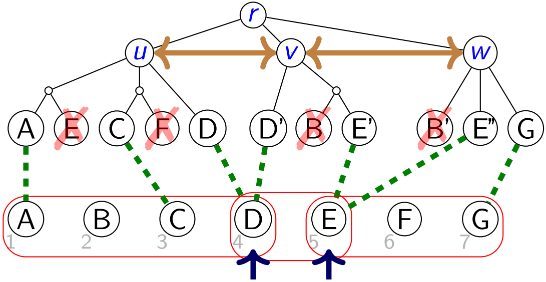

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Overall $O(d!n^{d+2})$

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OTDE is FPT for the deletion degree

From XP to FPT ($n^{f(d)} \rightarrow f(d)n^c$)

- augment the DP table with sets of children,
- progress one pivot at a time

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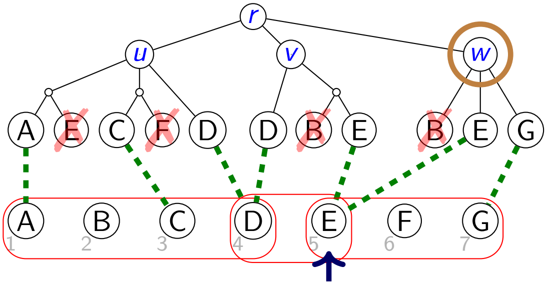

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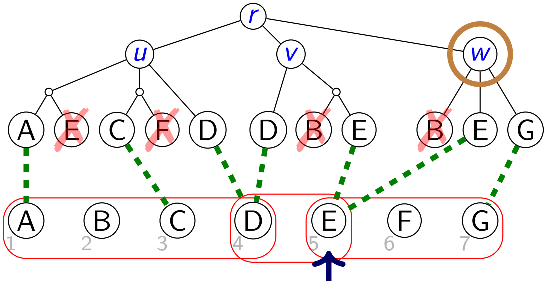

$$
\begin{aligned}
X(\{r\},1,7) &= X(\{u,v,w\},1,7) \\
&= X(\{u,v\},1,5) + X(\{w\},5,7)
\end{aligned}
$$

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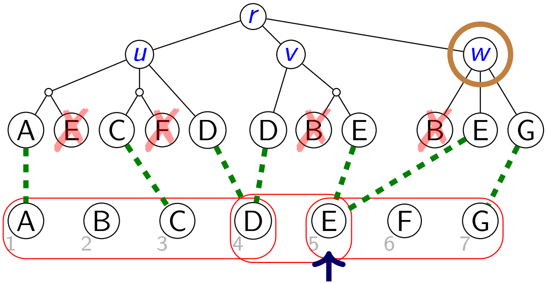

$$X(\{r\},1,7) = X(\{u,v,w\},1,7)$$
$$= X(\{u,v\},1,5) + X(\{w\},5,7)$$

Table size: $2^d n^3$, marginalization in $O(dn)$, overall: $O(d2^d n^4)$

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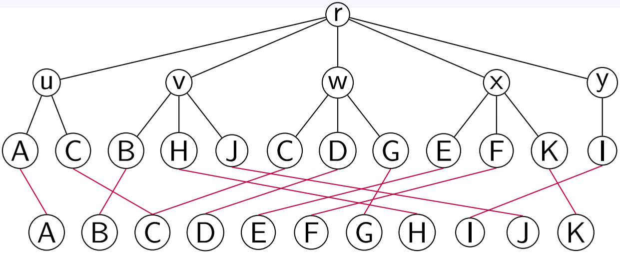
From degree to deletion-degree

- only $\partial \ll d$ children with a deletion
- there exists a large backbone without self-conflict



OTDE is FPT for the deletion degree

From degree to deletion-degree

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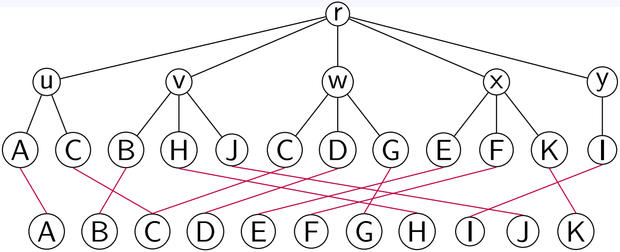
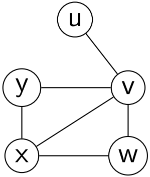
- compute some backbone using Vertex Cover





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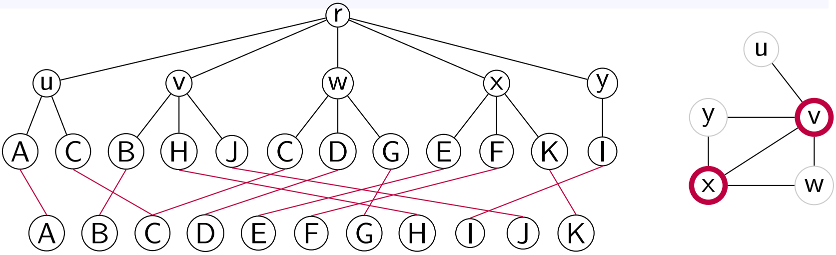
  VC$=\{x, v\} \to$ backbone $=(u, w, y)$



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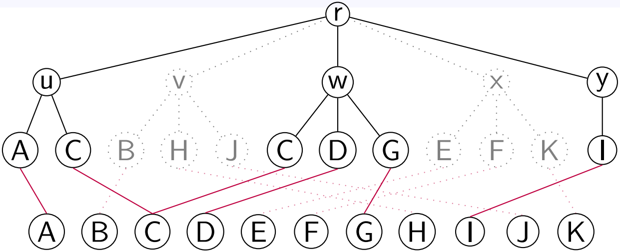

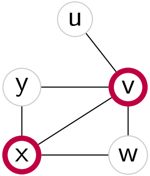

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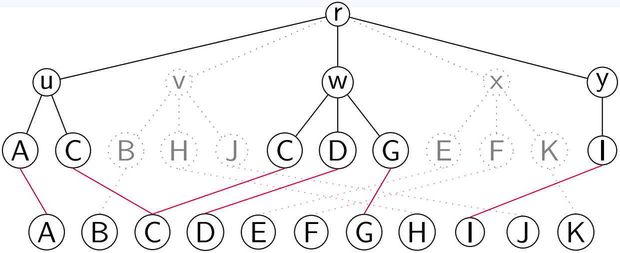
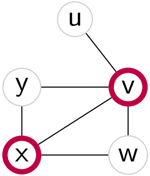
From degree to deletion-degree

- only $\partial \ll d$ children with a deletion
- there exists a large backbone without self-conflict
- compute some backbone using Vertex Cover
  VC$=\{x, v\} \to$ backbone $=(u, w, y)$
- compute DP entries for each
  (prefix of the backbone) $\cup$ (any vertices out of the backbone)





OTDE is FPT for the deletion degree

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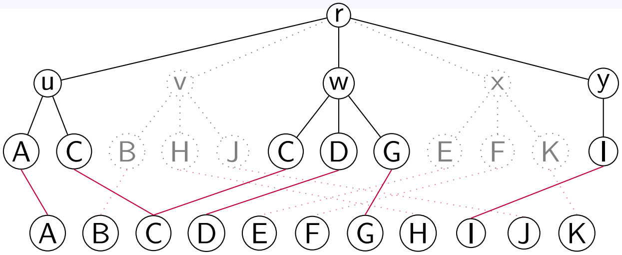
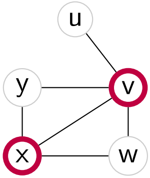
- $2^d \to d2^\partial$ (+ VC preprocessing in $O(1.3^\partial d + \partial d^2)$)





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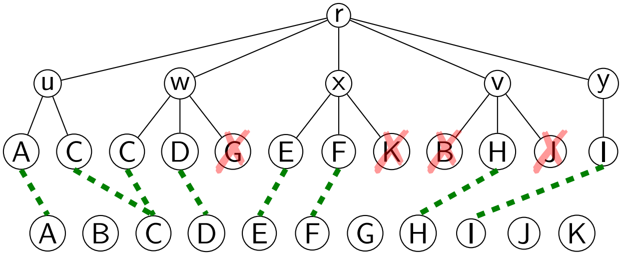
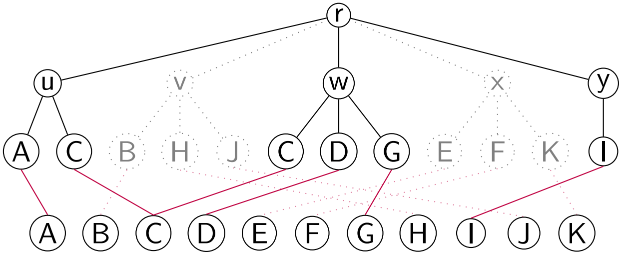
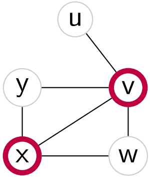


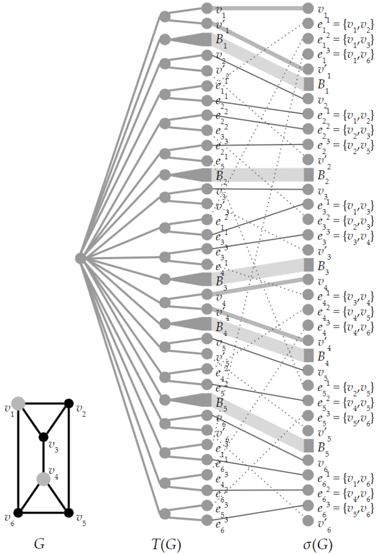




Hardness Results

OTDE is NP-hard

Reduction from Independent Set



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OTDE is NP-hard

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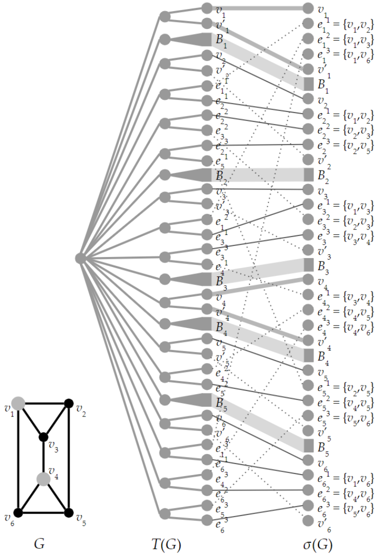
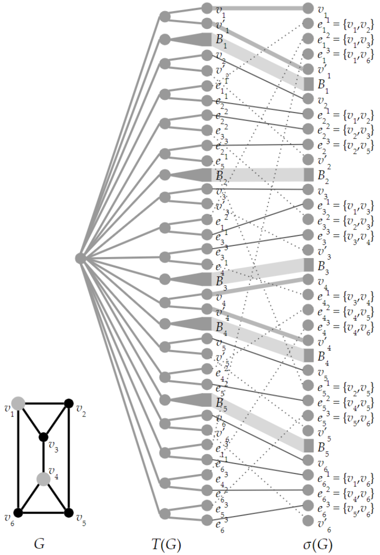


Given a graph G,
Build tree $T(G)$:

- One cherry per vertex (u, u')
- One cherry per edge (e, e')
- Separators

OTDE is NP-hard

Reduction from Independent Set



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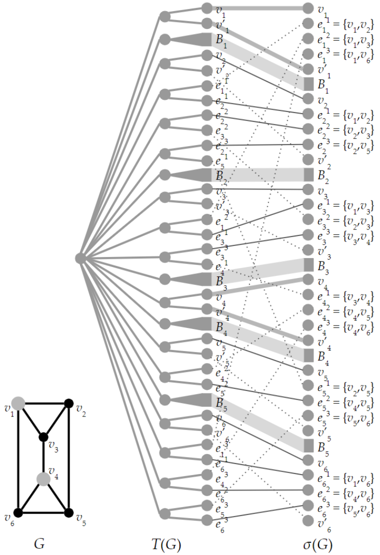
Build order $\sigma(G)$ (seen as a string):

- Factor $ue_1e_2e_3u'$ for each vertex and incident edges
- Separators between factors

OTDE is NP-hard

Reduction from Independent Set



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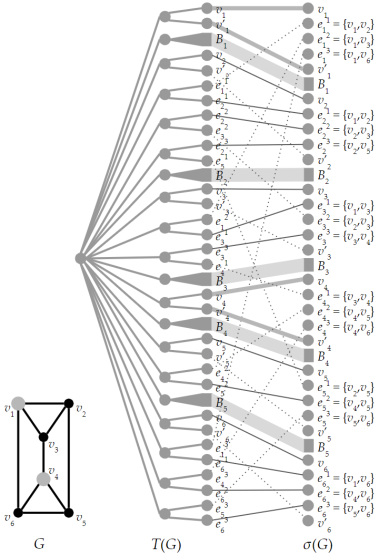

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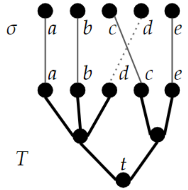
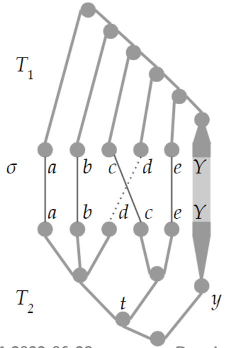
Wlog, delete ≤ 1 leaf per cherry, keep both leaves for vertices in an independent set.

TTDE is NP-hard

Reduction from OTDE

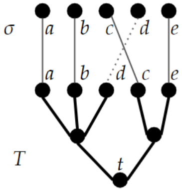


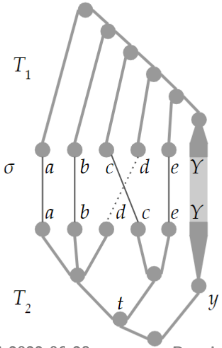
Given T, σ



TTDE is NP-hard

Reduction from OTDE



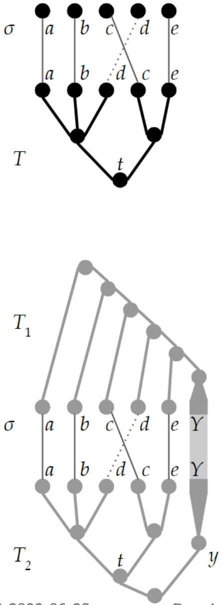


Given T, σ
Build T_1:

- Caterpillar following σ
- Large subtree ("anchor") at the bottom

TTDE is NP-hard

Reduction from OTDE



Given T, σ

Build T_1:

- Caterpillar following σ
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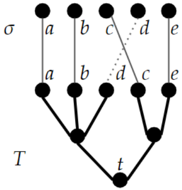
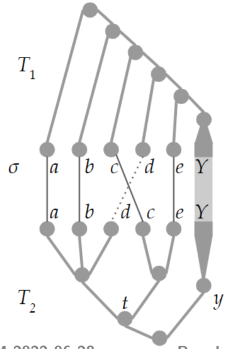
Build T_2:

- Start with T
- Connect anchor to the root

TTDE is NP-hard

Reduction from OTDE





Given T, σ

Build T_1:

- Caterpillar following σ
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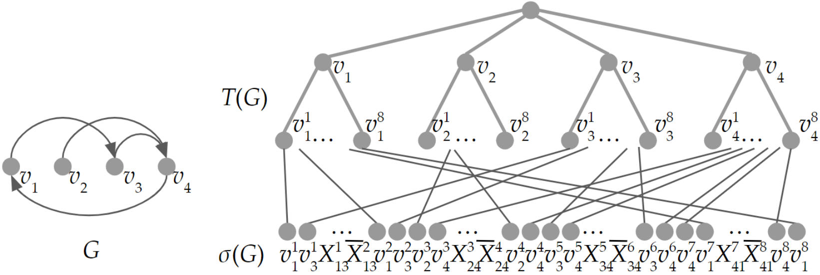
Build T_2:

- Start with T
- Connect anchor to the root

The anchor must be at one end of $T_1 \Rightarrow$ leaf order is the same as σ.

OTCM is NP-hard

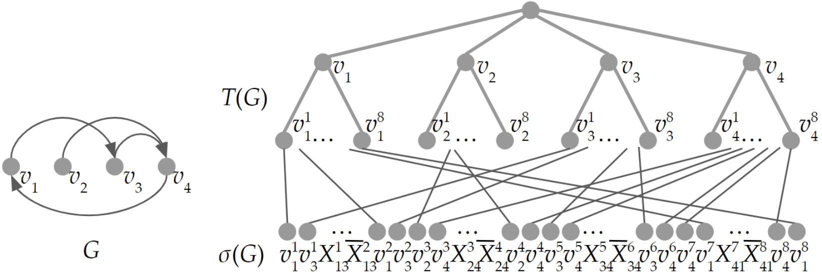
Reduction from Feedback Arc Set



Given G, build $T(G)$ with one large subtree per vertex.

OTCM is NP-hard

Reduction from Feedback Arc Set

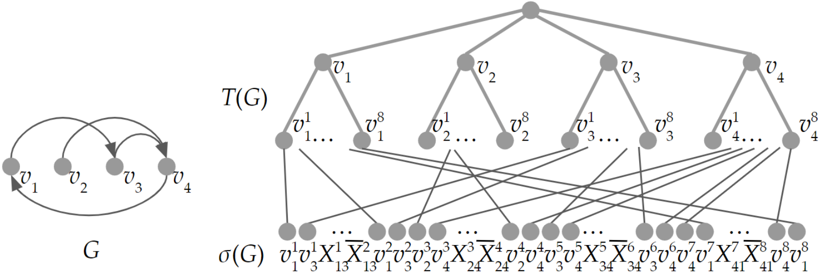


Given G, build $T(G)$ with one large subtree per vertex.
Build $\sigma(G)$ with one substring per arc :

$$v_1 \rightarrow v_3 \Longrightarrow v_1\, v_3\, v_2\, v_4\, v_4\, v_2\, v_1\, v_3$$

OTCM is NP-hard

Reduction from Feedback Arc Set



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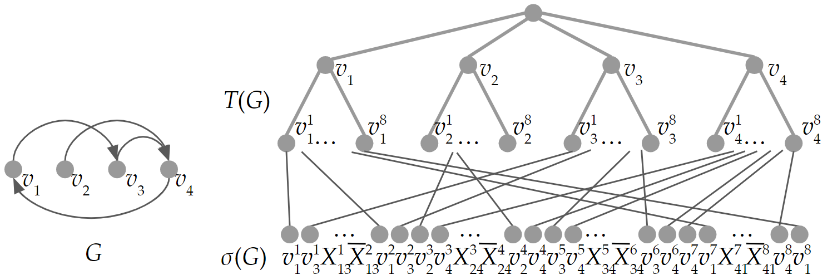
$$v_1 \rightarrow v_3 \implies v_1\, v_3\, v_2\, v_4\, v_4\, v_2\, v_1\, v_3$$

Solution: pick a permutation of the vertices

OTCM is NP-hard

Reduction from Feedback Arc Set



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Build $\sigma(G)$ with one substring per arc :

$$v_1 \to v_3 \Longrightarrow v_1 v_3 v_2 v_4 v_4 v_2 v_1 v_3$$

Solution: pick a permutation of the vertices
In the arc gadget:

- v_1, v_3 have 1 crossing if v_1 is before v_3, 3 otherwise
- Each other v_i, v_j have 2 crossings.

Experiments

Experiments: data & methods

Data

- Dated novels of 11 French 19$^{\text{th}}$ century writers
- Distance tables of novels using the relative frequencies of the 500 most frequent tokens
- Hierarchical clustering based on the distance tables, producing binary trees

Experiments: speed

tree	# leaves	OTCM time (ms)	# inversions	OTDE time (ms)	# deleted leaves
Ségur	22	1	40	200	9
Féval	23	2	47	268	8
Aimard	24	1	35	401	8
Zévaco	29	1	42	727	11
Lesueur	31	1	48	676	13
Zola	35	2	60	1203	9
Gréville	36	2	105	2211	18
Ponson	42	3	167	3447	18
Verne	58	3	183	13446	27
Balzac	59	4	248	8292	34
Sand	62	4	283	17557	39

Future work

$\Rightarrow$ Improve the dynamic programming algorithm/implementation solving OTDE

Experiments: presence of chronological signal

tree	# leaves	# inversions	p_{OTCM}	# deleted leaves	p_{OTDE}
Ségur	22	40	0.24	9	1
Féval	23	47	0.38	8	0
Aimard	24	35	0	8	0
Zévaco	29	42	0	11	0
Lesueur	31	48	0	13	0
Zola	35	60	0	9	0
Gréville	36	105	0	18	1
Ponson	42	167	2.23	18	0
Verne	58	183	0	27	0
Balzac	59	248	0	34	0
Sand	62	283	0	39	1

p_{OTCM}, p_{OTDE} = probablility (%) that a random order gives a better score than the chronological order

(over 10000 tries for OTCM, 100 for OTDE).

Experiments: identification of noise

Simulation experiment by adding errors in the leaf order

Repeat 100 times:

1. randomly choose "dates" from the interval [0,999]
2. build a distance matrix of the absolute differences between "dates" and the corresponding dendrogram
3. insert e artificial errors: pick a new random "date" for e randomly chosen leaves.

- Does OTDE output the set L_e of leaves with artificial errors?

Experiments: identification of noise

$n =$ # leaves	$e =$ # errors	proportion of cases when $L = L_e$	when $\lvert L - L_e \rvert = 1$
20	1	0.79	1
20	2	0.62	0.96
20	3	0.39	0.88
20	4	0.33	0.77
20	5	0.27	0.67
50	1	0.93	1
50	2	0.83	0.99
50	3	0.70	0.98
50	4	0.59	0.91
50	5	0.56	0.90

Observations

- if at most 2 errors, identified in more than 60% of the experiments, at least 1 identified in more than 96%.

Conclusion

Main results

- NP-hardness proofs for problems useful in bioinformatics and digital humanities
- FPT-algorithm in the deletion degree
- implementation in Python of an algorithm solving OTCM and OTDE, to evaluate the chronological signal in a tree
- a direct method to study the presence of the chronological signal in the data

Conclusion

Future works

- optimize the dynamic programming algorithm for OTDE
- evaluate the expected number of inversions or deleted leaves for a random order
- extend experiments:
  - run OTCM / OTDE on other datasets from different fields (see `https://github.com/oseminck/tree_order_evaluation`)
  - in-depth studies of cases where some leaves are expected to be wrongly ordered for OTDE
  - discuss the obtained results about the evolution of author styles with specialists of the authors