Permutation Pattern Matching for Double Partially Ordered Patterns

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Warm up game

Query

Find (if they exist) five cities, say A, B, C, D and E, such that:

- A and C are west of D and north of B,
- E is east of B and south of A,
- D is west of B and north of A and C.

Warm up game

Query

Find (if they exist) five cities, say A, B, C, D and E, such that:

- A and C are west of D and north of B,
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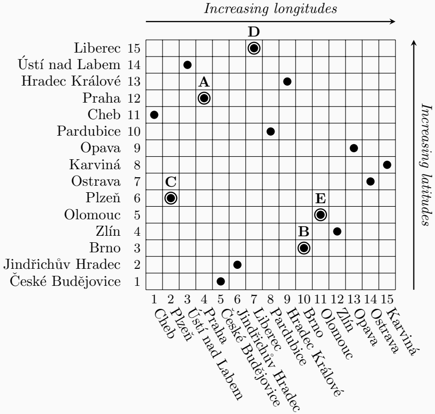
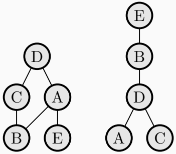
Warm up game (map as permutation)



$$\pi = 11\ 6\ 14\ 12\ 1\ 2\ 15\ 10\ 13\ 3\ 5\ 4\ 9\ 7\ 8$$

Warm up game (query as doubly ordered pattern)



P_v P_p

$P = (P_\mathrm{v}, P_\mathrm{p})$

	Cheb	Plzeň	Ústí nad Labem	Praha	České Budějovice	Jindřichův Hradec	Liberec	Pardubice	Hradec Králové	Brno	Olomouc	Zlín	Opava	Ostrava	Karviná
$\pi =$	11	6	14	12	1	2	15	10	13	3	5	4	9	7	8
	.	**C**	.	**A**	.	.	**D**	.	.	**B**	**E**	.	.	.	.
	A	**C**	**D**	.	.	**B**	.	.	.	**E**	.	.	.	.	.
	C	.	.	**A**	.	.	.	.	**D**	.	**B**	**E**	.	.	.

Query

Find (if they exist) five cities, say A, B, C, D and E, such that:

- A and C are west of D and north of B,
- E is east of B and south of A,
- D is west of B and north of A and C.

Permutation pattern

Pattern avoidance

A permutation σ occurs in another permutation π (or that π contains σ) if there exists a subsequence of elements of π that has the same relative order as σ.

Example

- π contains the permutation $\sigma = 123$ (resp $\sigma = 321$) if it has an increasing (resp a decreasing) subsequence of size 3.
- $\sigma = 4312$ occurs in $\pi = 6152347$, as shown in 6152347.
- $\pi = 6152347$ avoids $\sigma' = 2341$.

Permutation pattern

Permutation Pattern Matching

PERMUTATION PATTERN MATCHING is the problem to decide is some permutation σ occurs in another permutation π.

- **PERMUTATION PATTERN MATCHING** is NP-hard [Bose, Buss, Lubiw. 98].
- **PERMUTATION PATTERN MATCHING** is FPT [Guillemot, Marx. 14].
- (Relatively) Fast exponential-time algorithms [Gawrychowski, Rzepecki. 22 ; Berendsohn, Kozma, Marx. 21].

Permutation pattern

Permutation Pattern Matching

PERMUTATION PATTERN MATCHING is the problem to decide is some permutation σ occurs in another permutation π.

- **PERMUTATION PATTERN MATCHING** is solvable in $O(|\sigma|^4)$ time if σ is $(2413, 3142)$-avoiding (aka. *separable*) [Ibarra. 95 ; Neou, Rizzi, V. 16].
- **PERMUTATION PATTERN MATCHING** is in P if both π and σ are 321-avoiding [Guillemot, V. 09 ; Albert, Lackner, Lackner, Vatter. 16].
- **PERMUTATION PATTERN MATCHING** is NP-complete if π is 4321-avoiding and σ is 321-avoiding [Jelínek, Kynčl. 17].
- **PERMUTATION PATTERN MATCHING** is linear-time solvable if both π and σ are $(213, 231)$-avoiding [Neou, Rizzi, V. 16].
- ...

Permutation pattern

Generalizations

- Consecutive patterns [Elizalde, Noy. 03].
- Barred patterns [West. 90].
- Vincular (or generalized) patterns [Babson, Steingrímsson. 00].
- Bivincular patterns [Bousquet-Mélou, Claesson, Dukes, Kitaev. 10].
- Partially ordered patterns [Kitaev. 07].
- Parity permutation matching [Ardévol Martínez, Sikora, V. 22].
- ...

Vincular patterns

Definition

A vincular pattern σ of length k is a permutation in $\mathfrak{S}(k)$ some of whose consecutive letters may be underlined.

If σ contains $\underline{\sigma_i \sigma_{i+1} \ldots \sigma_j}$ then the letters corresponding to $\sigma_i \sigma_{i+1} \ldots \sigma_j$ in an occurrence of σ in a permutation π must be adjacent.

if σ begins with ⚓σ_1 then any occurrence of σ in a permutation π must begin with the leftmost element of π.

if σ ends with σ_k⚓ then any occurrence of σ in π must end with the rightmost element of π.

Vincular patterns

Examples

σ	Occurrences in $\pi = 241563$
231	241, 453, 463 and 563
$\underline{23}1$	241 and 563
$2\underline{31}$	241, 463 and 563
⚓$\underline{231}$	241
$\underline{231}$⚓	563
231⚓	453, 463 and 563

Partially ordered patterns

(Informal) Definition

The idea of partially ordered patterns is to allow the possibility for some letters to be incomparable in the sense that we do not know the relative order of these letters.

Partially ordered patterns

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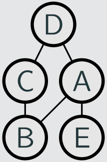
Examples



if $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_4}\pi_{i_5}$ is an occurrence of $\sigma =$ ⚓$B\underline{AC}ED$ in some permutation π, then $i_1 = 1$, $i_3 = i_2 + 1$, $\pi_{i_1} < \pi_{i_2}$ and $\pi_{i_1} < \pi_{i_3}$, $\pi_{i_2} < \pi_{i_5}$, $\pi_{i_3} < \pi_{i_5}$ and $\pi_{i_4} < \pi_{i_2}$.

Doubly ordered pattern

dpop

A doubly partially ordered pattern (dpop) is a pair $P = (P_{\mathsf{v}}, P_{\mathsf{p}})$, of posets $P_{\mathsf{v}} = (X, \leqslant_{\mathsf{v}})$ and $P_{\mathsf{p}} = (X, \leqslant_{\mathsf{p}})$ defined over the same set X.

We call P_{v} and P_{p} the value poset and the position poset, respectively.

dpop (cont.)

A dpop $P = (P_{\mathsf{v}}, P_{\mathsf{p}})$ is

- symmetric if $P_{\mathsf{v}} = P_{\mathsf{p}}$,
- dual if $P_{\mathsf{v}} = {P_{\mathsf{p}}}^{\partial}$, and
- semi-total if one of P_{p} or P_{v} is a total order.

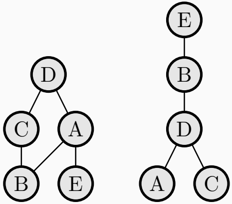
Doubly ordered pattern

DPOP MATCHING

Given a permutation $\pi \in \mathfrak{S}(n)$ and a dpop $P = (P_\mathsf{v}, P_\mathsf{p})$, an occurrence (or mapping) of P in π is an injective function $\varphi : X \to [n]$ such that:

- $\pi \circ \varphi$ is $\leqslant_\mathsf{v}$-non-decreasing, i.e., for all $x, y \in X$, if $x \leqslant_\mathsf{v} y$ then $\pi(\varphi(x)) \leqslant \pi(\varphi(y))$, and
- φ is $\leqslant_\mathsf{p}$-non-decreasing, i.e., for all $x, y \in X$, if $x \leqslant_\mathsf{p} y$ then $\varphi(x) \leqslant \varphi(y)$.

DPOP MATCHING is the problem to decide whether P occurs in π.



P_{v} P_{p}

$P = (P_{\mathrm{v}}, P_{\mathrm{p}})$

	Cheb	Plzeň	Ústí nad Labem	Praha	České Budějovice	Jindřichův Hradec	Liberec	Pardubice	Hradec Králové	Brno	Olomouc	Zlín	Opava	Ostrava	Karviná
$\pi =$	11	6	14	12	1	2	15	10	13	3	5	4	9	7	8
	.	**C**	.	**A**	.	.	**D**	.	.	**B**	**E**	.	.	.	.
	A	**C**	**D**	.	.	**B**	.	.	.	**E**	.	.	.	.	.
	C	.	.	**A**	.	.	.	.	**D**	.	**B**	**E**	.	.	.

Doubly ordered pattern

Observation

PERMUTATION PATTERN MATCHING is the special case of **DPOP MATCHING** where both $\leqslant_{\mathrm{v}}$ and $\leqslant_{\mathrm{p}}$ are total orders, and hence **DPOP MATCHING** is NP-hard.

Observation

Let $P = (P_{\mathrm{v}}, P_{\mathrm{p}})$ be a dpop and π be a permutation. The following statements are equivalent:

1. $(P_{\mathrm{v}}, P_{\mathrm{p}})$ occurs in π;
2. $(P_{\mathrm{v}}, (P_{\mathrm{p}})^{\partial})$ occurs in π^{r};
3. $((P_{\mathrm{v}})^{\partial}, P_{\mathrm{p}})$ occurs in π^{c};
4. $((P_{\mathrm{v}})^{\partial}, (P_{\mathrm{p}})^{\partial})$ occurs in $\pi^{\mathrm{c\,r}}$;
5. $(P_{\mathrm{p}}, P_{\mathrm{v}})$ occurs in π^{-1}.

Doubly ordered pattern

Observation

Let $P = (P_\mathsf{v}, P_\mathsf{p})$ be a dpop with $P_\mathsf{v} = (X, \leqslant_\mathsf{v})$, $P_\mathsf{p} = (X, \leqslant_\mathsf{p})$ and $k = |X|$, and let $\pi \in \mathfrak{S}(n)$ be a permutation. The following statements are equivalent:

- P occurs in π.
- There exists a linear extension $\tau_\mathsf{v} : X \to [k]$ of P_v and a linear extension $\tau_\mathsf{p} : X \to [k]$ of P_p such that the permutation $\sigma \in \mathfrak{S}(k)$ defined by $\sigma(i) = \tau_\mathsf{v}(\tau_\mathsf{p}^{-1}(i))$ for $1 \leqslant i \leqslant k$ is contained in π.

Corollary

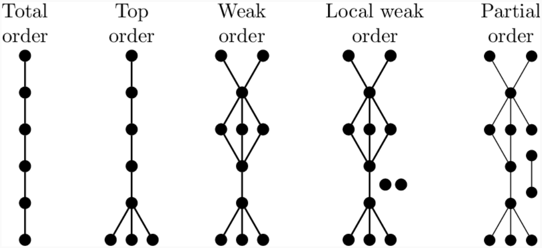
DPOP MATCHING *is* FPT *for the parameter* $|P|$*.*

Weak order

A partial order $\leqslant$ is a on a set X is a weak order if the following property holds:

for all pairwise distinct $x, x', x'' \in X$, if x is incomparable with x' (i.e., neither $x \leqslant x'$ nor $x' \leqslant x$ is true) and if x' is incomparable with x'', then x is incomparable with x''.

A partial order is *k*-weak if it the disjoint union of k weak orders.



Semi-total patterns

Recall that a dpop $P = (P_\mathrm{v}, P_\mathrm{p})$ is semi-total if one of P_p or P_v is a total order.

Theorem

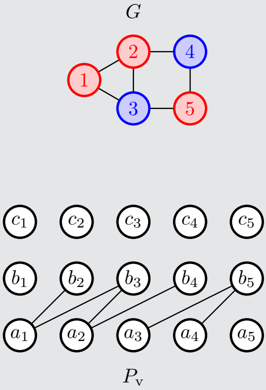
DPOP MATCHING *is solvable in* $O\left(n^{\mathrm{height}(P_\mathrm{v})}\right)$ *time if* P_v *is a weak order and* P_p *is a total order.*

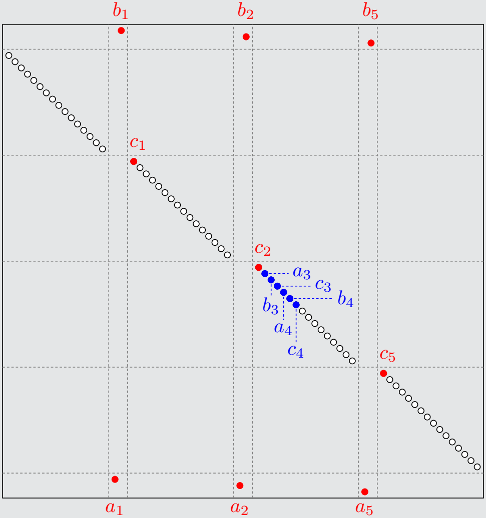
Theorem

DPOP MATCHING *is* NP*-complete even if* $\mathrm{height}(P_\mathrm{v}) = 2$, P_p *is a total order and* π *avoids* 1234.

Semi-total patterns

Proof elements





Symmetric patterns & bounded width

Recall that a dpop $P = (P_\mathsf{v}, P_\mathsf{p})$ is symmetric if $P_\mathsf{v} = P_\mathsf{p}$.

Theorem ([Crochemore, Porat. 10])

DPOP MATCHING *is solvable in* $O(n \log\log |P|)$ *time if* P *is a symmetric dpop of width 1 (i.e., a total symmetric dpop* P*).*

Symmetric patterns & bounded width

BALANCED k-INCREASING COLORING

Given a positive integer k and a permutation π, **BALANCED k-INCREASING COLORING** is the problem of deciding whether there exists a balanced k-coloring of π (i.e., a partition of $[kn]$ into k subsets of size exactly n) such that each color induces an increasing subsequence of π.

Theorem

BALANCED k-INCREASING COLORING *for* 312*-avoiding permutations is* W*[1]-hard for the parameter k.*

Symmetric patterns & bounded width

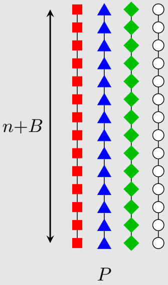
Proof elements

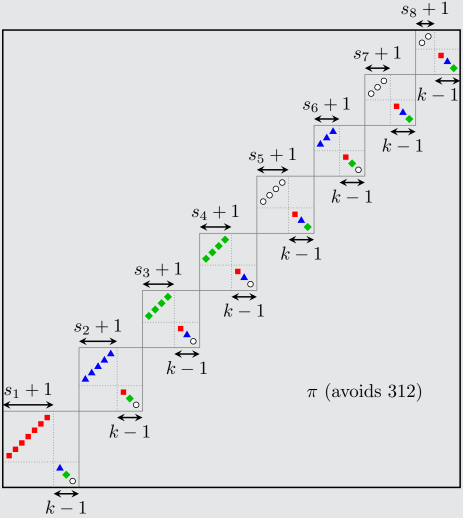
Reduction from **UNARY BIN PACKING** which is known to be is W[1]-hard [Jansen, Kratsch, Marx. 13].

Given a set of integers $S = \{s_1, s_2, \ldots, s_n\}$ encoded in unary and two positive integers B and k, decide whether S can be partitioned into k subsets, each of total size B.

Symmetric patterns & bounded width

Proof elements





Symmetric patterns & bounded width

Proof elements

$(\Rightarrow)$

$\sum_{s_i \in S_j} (s_i + 1) + n - |S_i| = \sum_{s_i \in S_j} s_i + |S_i| + n - |S_i| = B + n.$

$(\Leftarrow)$

Every gadget requires k colors, and hence the increasing part of each gadget is monochromatic.

$$\begin{aligned} B + n &= \sum_{s_i \in S_j} (s_i + 1) + n - |S_j| \\ &= \sum_{s_i \in S_j} s_i + |S_j| + n - |S_j| \\ &= \sum_{s_i \in S_j} s_i + n. \end{aligned}$$

Symmetric patterns & bounded width

Theorem

DPOP MATCHING *for symmetric dpop and* 312*-avoiding permutations is* W*[1]-hard for the parameter* width(P)*.*

Proof elements

Reduce from **BALANCED** k**-INCREASING COLORING** for 312-avoiding permutations.

Construct a symmetric dpop $P = (\mathcal{P}, \mathcal{P})$, where $\mathcal{P} = (X, \preccurlyeq)$, as follows: $X = [k] \times [n]$ and $(i, j) \preccurlyeq (i', j')$ if and only if $i = i'$ and $j \leqslant j'$.

We claim that P occurs in π if and only π admits a k-coloring for which every color induces an increasing pattern of length n.

Symmetric patterns & bounded width

Theorem

DPOP MATCHING *for k-weak symmetric dpop is* XP *with parameter k.*

Symmetric patterns & pattern avoidance

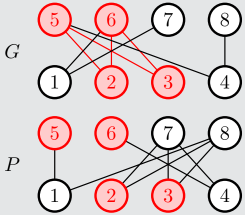
Theorem

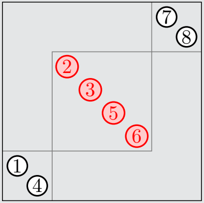
DPOP MATCHING *for height-2 symmetric dpop P and permutation π is* NP*-hard even if π is separable (it avoids* 2413 *and* 3142*) and one of the following restrictions occurs:*

1. π *is* 123*-avoiding;*
2. π *is* $(132, 213)$*-avoiding;*
3. π *is* $(132, 321)$*-avoiding;*
4. π *is* $(231, 312)$*-avoiding;*
5. π *is* $(132, 312)$*-avoiding;*
6. π *is* $(213, 321)$*-avoiding;*
7. π *is* $(213, 312)$*-avoiding;*
8. π *is* $(132, 231)$*-avoiding;*
9. π *is* $(213, 231)$*-avoiding.*

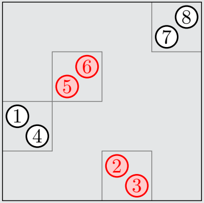
π is $(231, 312)$-avoiding.

Proof elements





π_1 (avoids 231,312)



π_2 (avoids 132,312)

Symmetric patterns & pattern avoidance

Theorem

DPOP MATCHING *is in* P *for symmetric dpop P if one of the following restrictions on π occurs:*

1. π *is* $(123, 231)$*-avoiding;*
2. π *is* $(123, 132)$*-avoiding;*
3. π *is* $(123, 321)$*-avoiding;*
4. π *is* $(123, 312)$*-avoiding;*
5. π *is* $(123, 213)$*-avoiding.*

Symmetric patterns & pattern avoidance

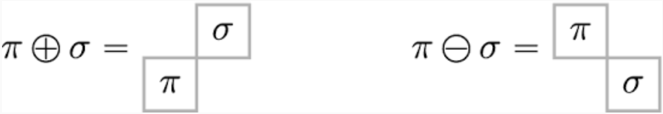
direct sum, $\pi \in \mathfrak{S}(m)$, $\sigma \in \mathfrak{S}(n)$

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{for } 1 \leqslant i \leqslant m, \\ \sigma(i-m)+m & \text{for } m+1 \leqslant i \leqslant m+n. \end{cases}$$

skew sum, $\pi \in \mathfrak{S}(m)$, $\sigma \in \mathfrak{S}(n)$

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i)+n & \text{for } 1 \leqslant i \leqslant m, \\ \sigma(i-m) & \text{for } m+1 \leqslant i \leqslant m+n, \end{cases}$$

and



Symmetric patterns & pattern avoidance

π is $(123, 231)$-avoiding

Proof elements

There exist integers k, ℓ and m, with sum n, such that $\pi = \mathrm{dp}(k) \ominus (\mathrm{dp}(\ell) \oplus \mathrm{dp}(m))$.

π is $(123, 132)$-avoiding

Proof elements

The permutation π is a skew sum $\pi = \ominus_{p=1}^{k} d_p$ of patterns of the form $d_p = \mathrm{dp}(a_p) \oplus \mathrm{dp}(1)$ for some integer $a_p \geqslant 0$.

Symmetric patterns & pattern avoidance

π avoids … and …	123	132	213	231	312	321
123	**3P**					
132						
213		**BIN**				
231		**BIC**	**BIC**			
312		**BIC**	**BIC**	**BIC**		
321		**BIS**	**BIS**	(open)		

Poynomial cases

NP-hard cases

BIC = Biclique,

3P = 3-Partition,

BIN = Unary Bin Packing,

BIS = Bisection.

Concluding remarks

General remarks

- Generate instances that yield a unique solution (minimizing $|X|$).
- Permutation classes.

Semi-total patterns

- Constant width patterns.
- Most classes of pattern avoiding permutations.

Symmetric patterns

- $(231, 321)$-avoiding patterns.
- $(312, 321)$-avoiding patterns.