On Strings Having the Same Length-k Substrings

CPM 2022

Giulia Bernardini[1,5], Alessio Conte[2], **Estéban Gabory**[1], Roberto Grossi[2], Grigorios Loukides[3], Solon P. Pissis[1,4], Giulia Punzi[2] and Michelle Sweering[1]

[1]CWI Amsterdam, [2]Università di Pisa, [3]King's College London, [4]Vrije Universiteit Amsterdam, [5]Università di Trieste

June 2022

Outline

Introduction and preliminaries

DCP and Shortest $\mathcal{S}$-Equivalent String

Combinatorial bounds

Solving z-Shortest $\mathcal{S}$-Equivalent String

Shortest equivalent string : problem definition

SHORTEST $\mathcal{S}$-EQUIVALENT STRING
Input: A set $\mathcal{S}$ of n length-k strings.
Output: A shortest string T such that the set of length-k substrings of T is $\mathcal{S}$, or FAIL if that is not possible.

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Generalization z-SHORTEST $\mathcal{S}$-EQUIVALENT STRING: We want the z shortest strings satisfying the property, in increasing length order.

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  - The length k strings are k-mers in a genome.
  - Find a shorter string having a given *k-mer spectrum*.

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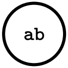

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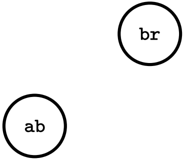

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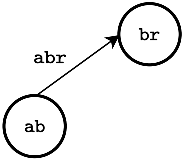

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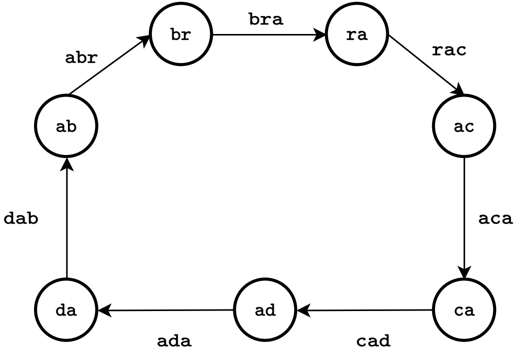

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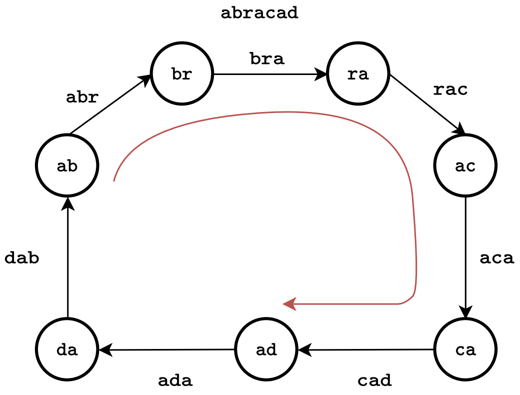

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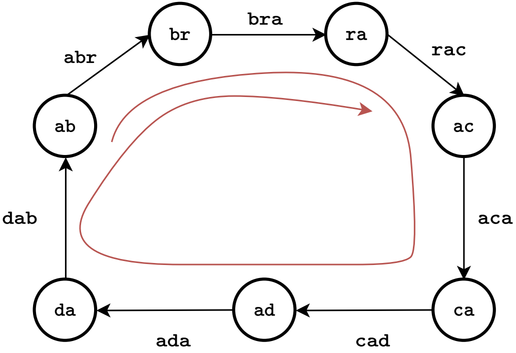

Eulerian walks

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The strings having the set of their length-k substrings *included in* $\mathcal{S}$ correspond to walks on the de Bruijn graph of $\mathcal{S}$.

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Definition

An *Eulerian walk* on a graph G is a walk on G that traverses every edge *at least* once.

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The strings having the set of their length-k substrings *equal to* $\mathcal{S}$ correspond to *Eulerian* walks on the de Bruijn graph of $\mathcal{S}$.

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Eulerian walks

Consequence : To solve Shortest $\mathcal{S}$-Equivalent String and z-Shortest $\mathcal{S}$-Equivalent String, we need to find Eulerian walks in De Bruijn graphs.

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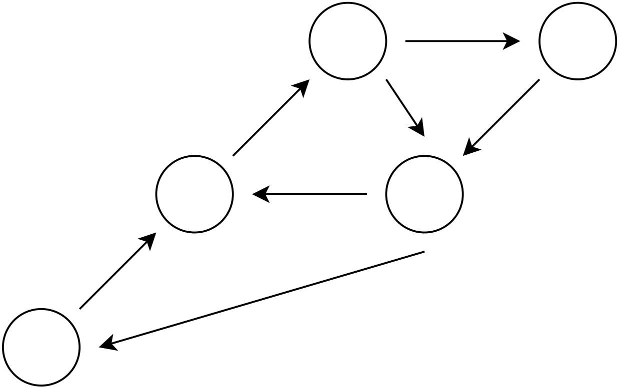

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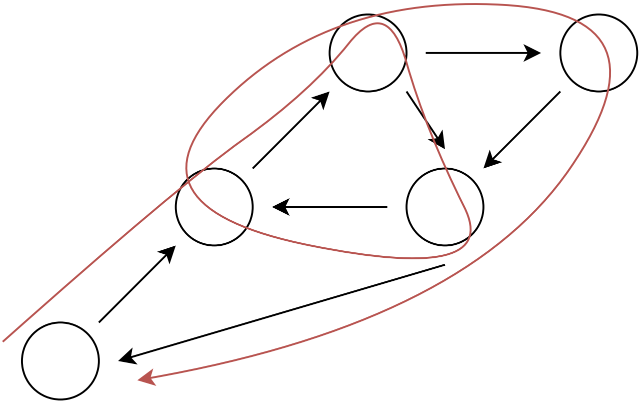

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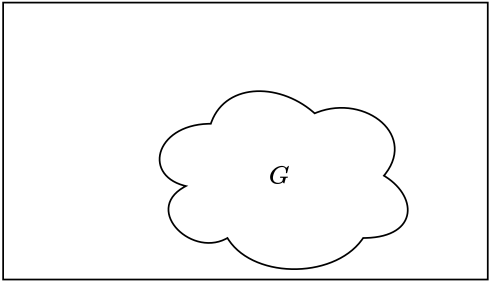

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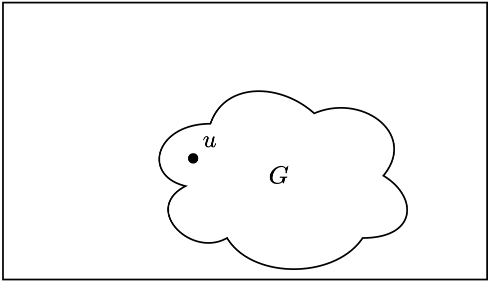

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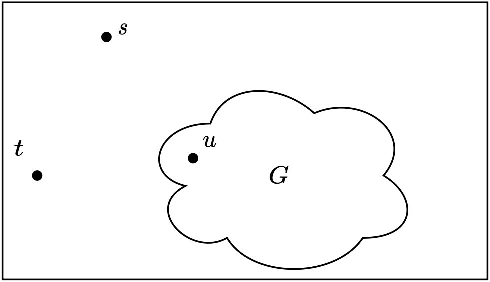

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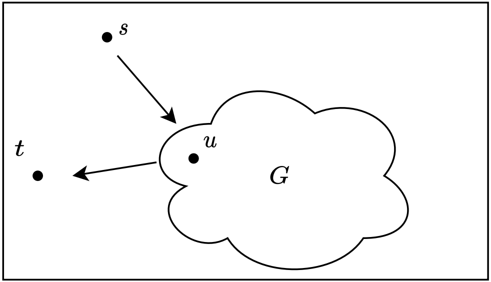

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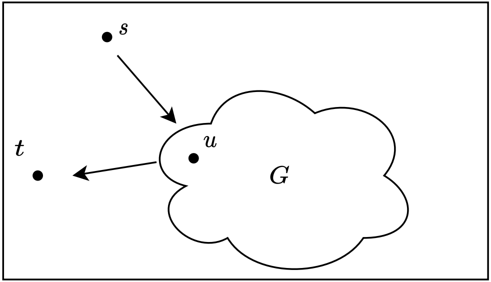
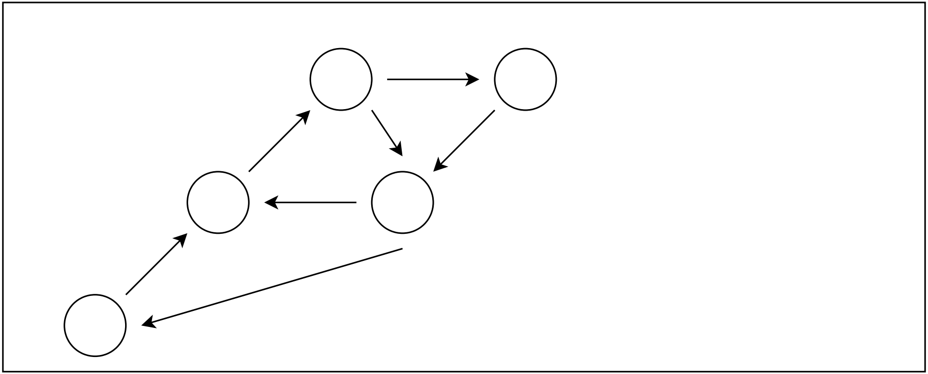


One can find the closed Eulerian walks on G by finding the open Eulerian walks on the extended graph.

Reducing DIRECTED CHINESE POSTMAN to SHORTEST $\mathcal{S}$-EQUIVALENT STRING (large alphabet)



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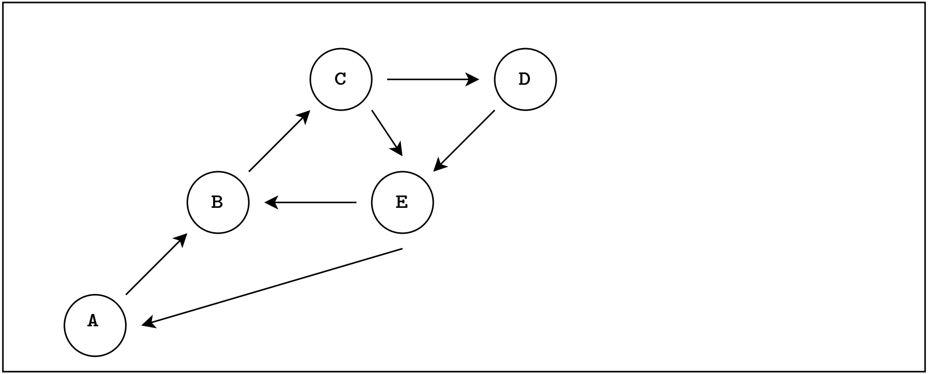

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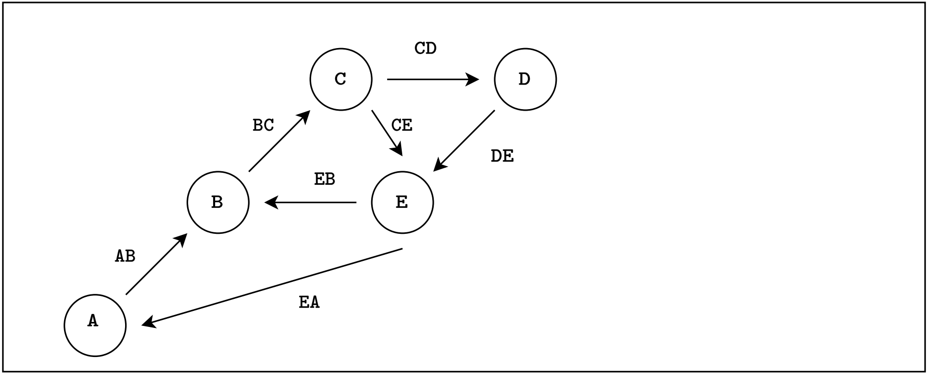

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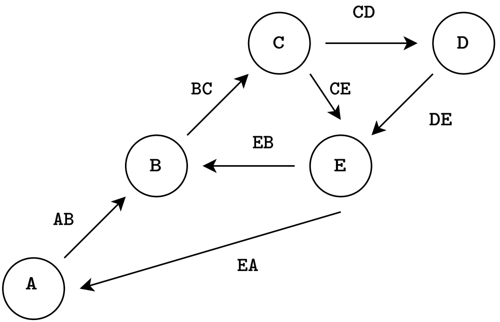

$$\left\{ \begin{array}{c} \mathtt{AB}, \mathtt{EA}, \\ \mathtt{EB}, \mathtt{BC}, \\ \mathtt{CD}, \mathtt{DE} \\ \mathtt{CE} \end{array} \right\}$$

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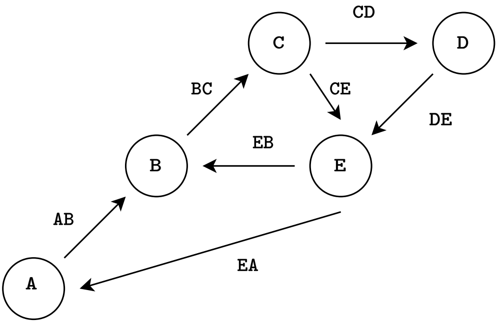

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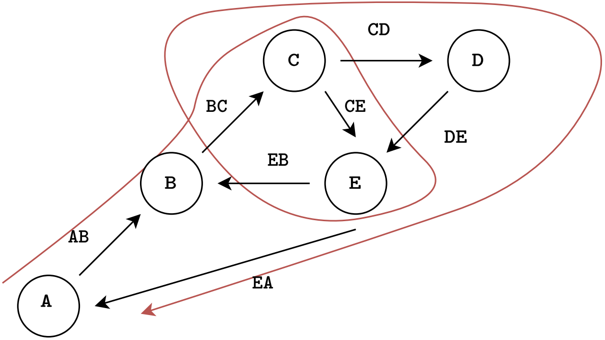

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Reducing Directed Chinese Postman to Shortest $\mathcal{S}$-Equivalent String (large alphabet)

Theorem

Any instance of Directed Chinese Postman *can be reduced to an instance of* Shortest $\mathcal{S}$-Equivalent String *in linear time with* $||\mathcal{S}|| = \mathcal{O}(|E|)$.

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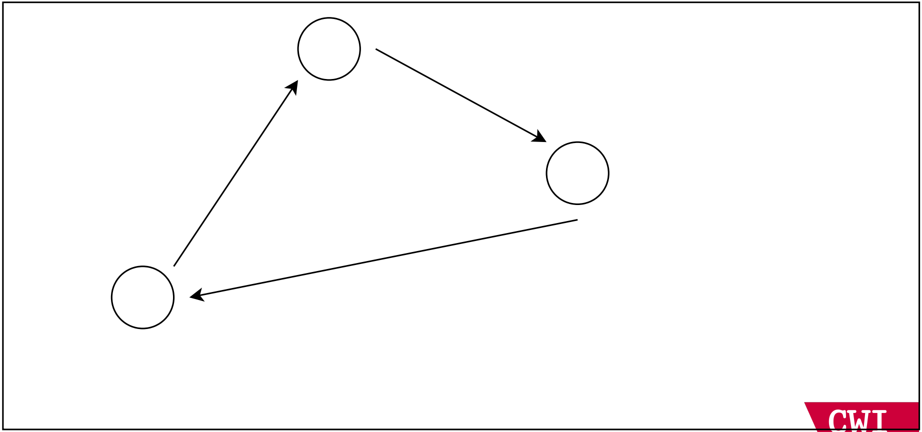

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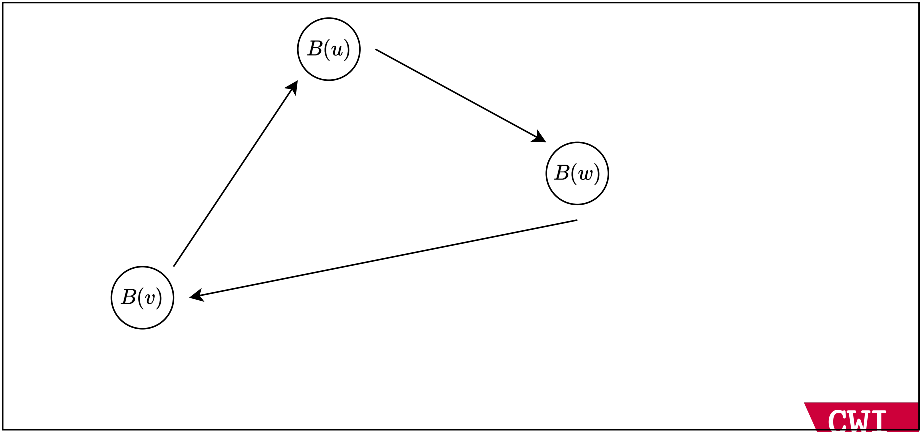

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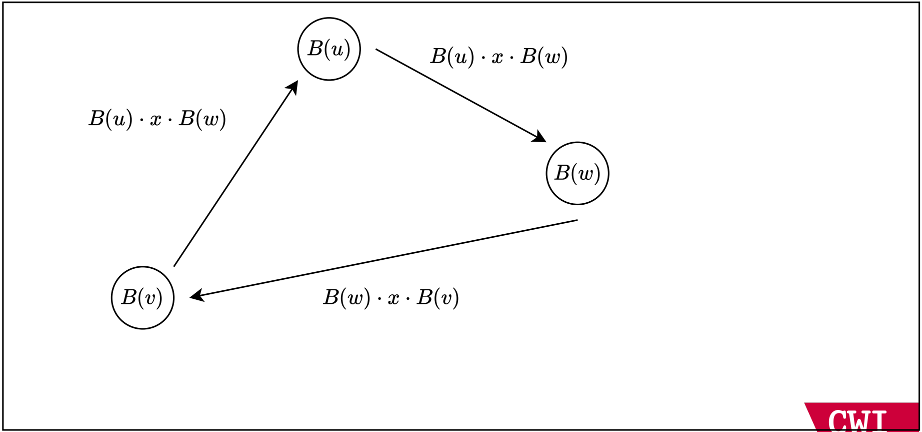

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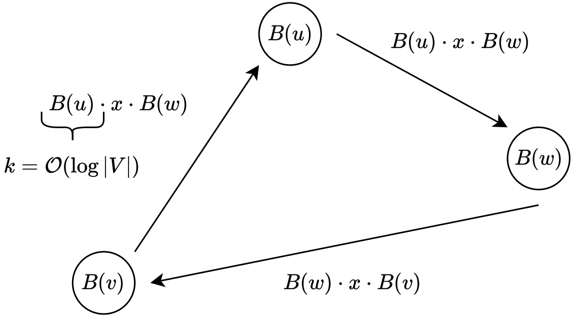

x is a fixed string having length $\mathcal{O}(\log |V|)$

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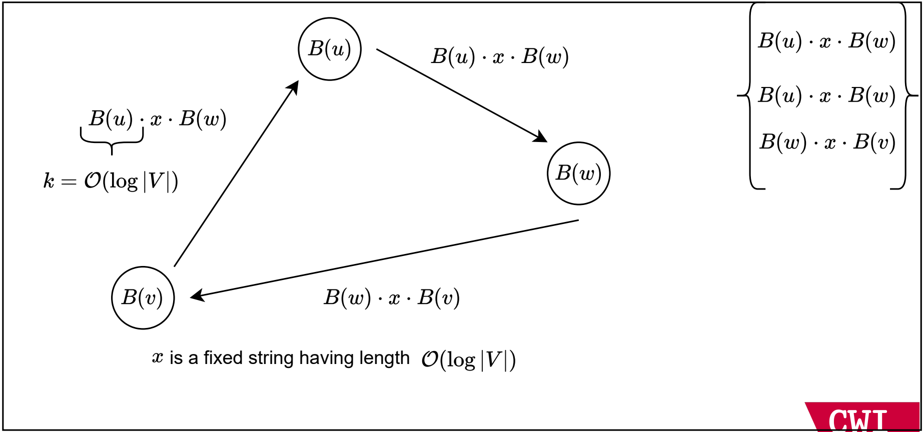

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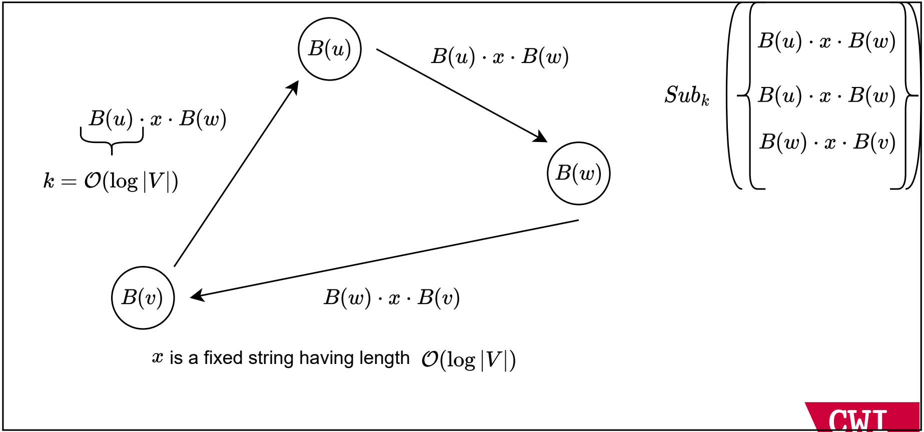

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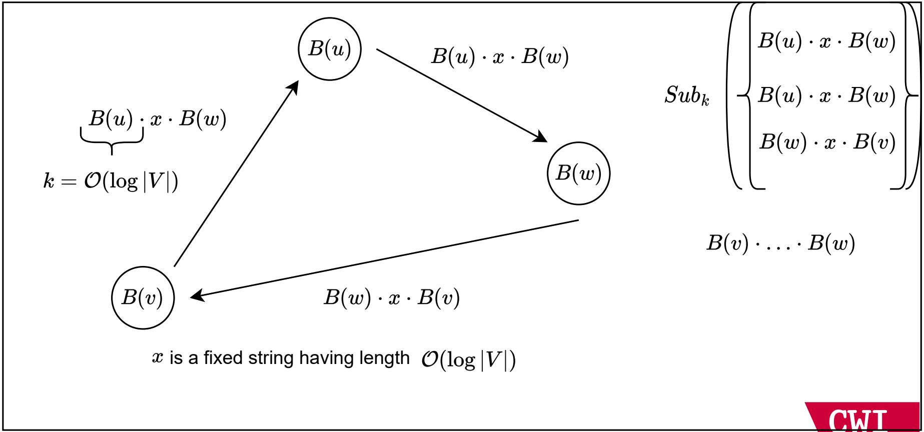

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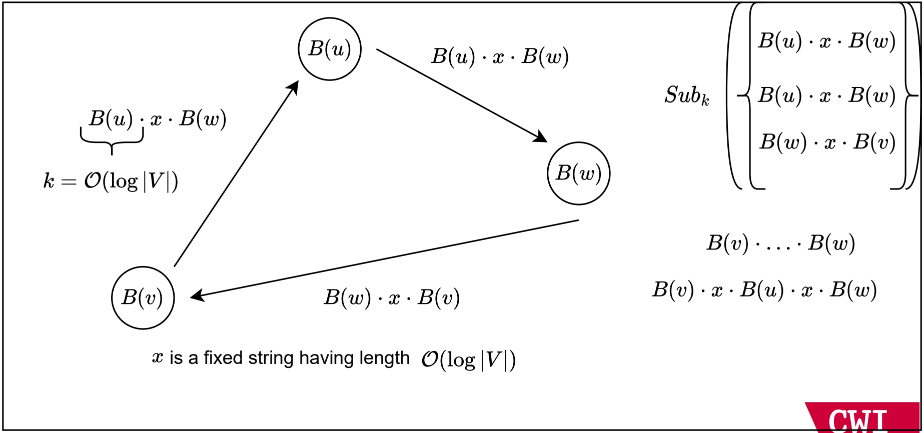

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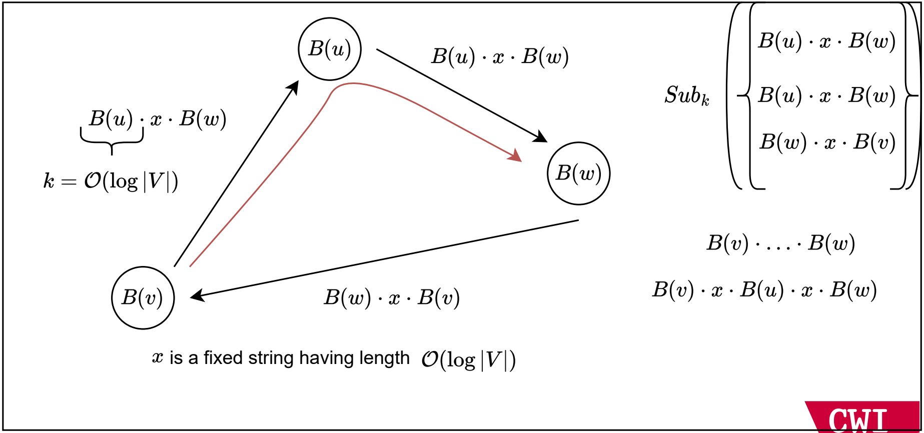

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Any instance $G(V, E)$ of Directed Chinese Postman *with output $\mathcal{W}$ can be reduced in $\mathcal{O}(|E| \log |V| + |\mathcal{W}|)$ time to an instance of* Shortest $\mathcal{S}$-Equivalent String *on a binary alphabet, where $\mathcal{S}$ is a set of $\mathcal{O}(|E| \log |V|)$ strings having length $\log |V|$.*

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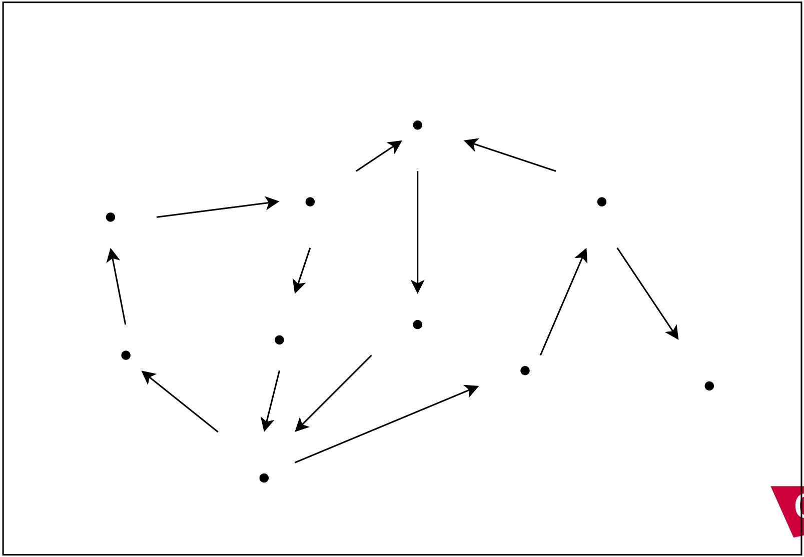
Consequence : If the Shortest $\mathcal{S}$-Equivalent String problem over a *binary alphabet* has a near-linear-time solution then so does Directed Chinese Postman.

Outline

Length of the shortest Eulerian walk on a graph



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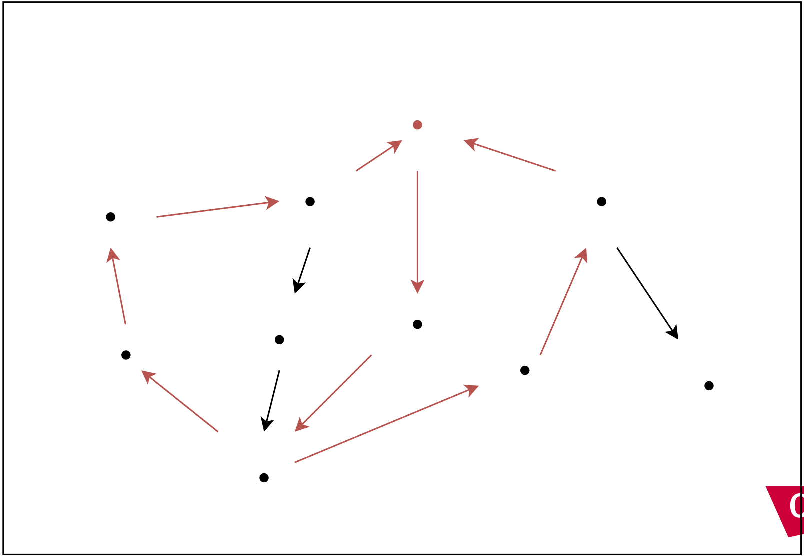

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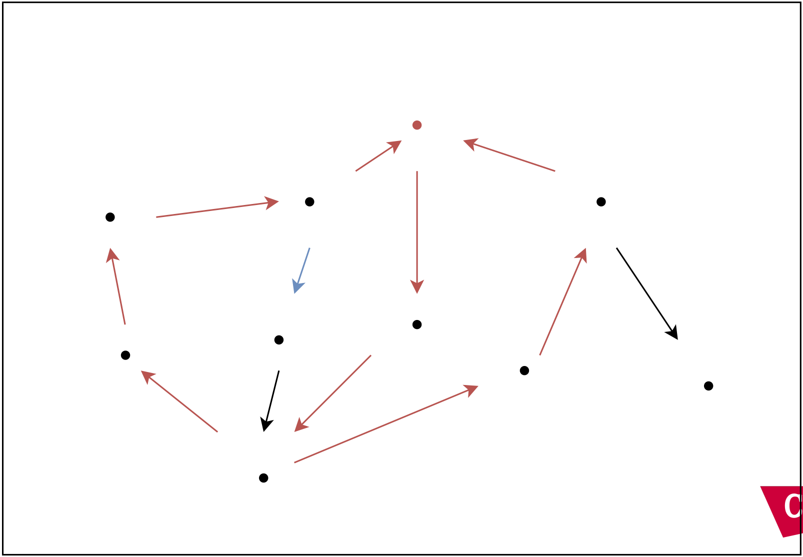

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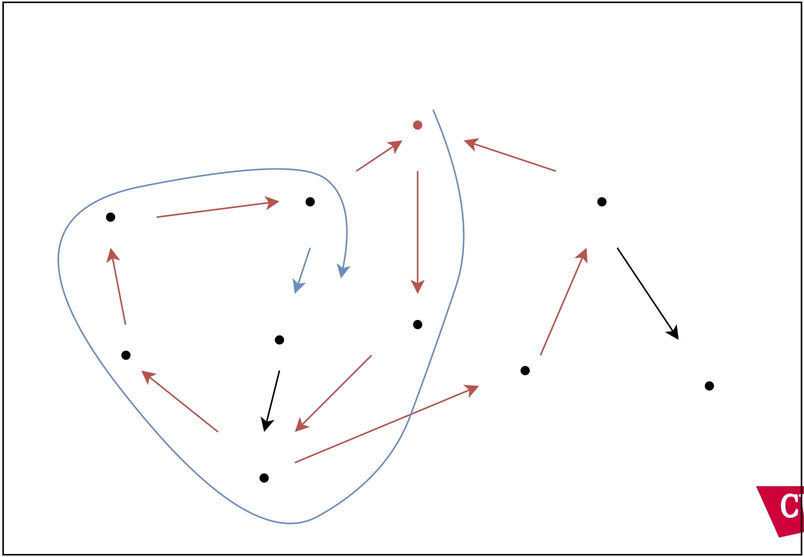

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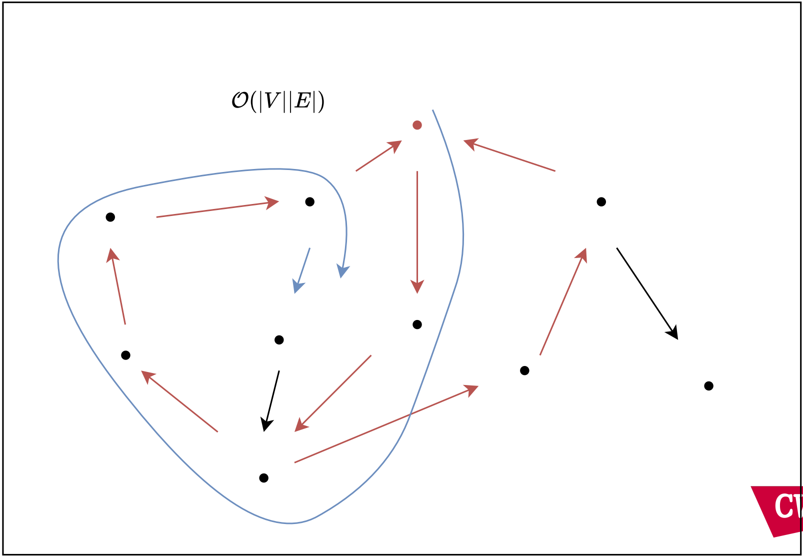

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Theorem

If $\mathcal{W}$ is the shortest Eulerian walk on a graph $G(V, E)$, then $|\mathcal{W}| = \mathcal{O}(|V|\,|E|)$.

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If $\mathcal{W}$ is the shortest Eulerian walk on a graph $G(V, E)$, then $|\mathcal{W}| = \mathcal{O}(|V|\,|E|)$ and this bound is asymptotically tight.

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Lemma

Let $\mathcal{S}$ be a set containing n strings of length k each. Then the De Bruijn graph of $\mathcal{S}$ has at most $n+1$ nodes and n edges. In this graph, a walk $\mathcal{W}$ corresponds to a string of length $|\mathcal{W}| + k - 1$.

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Theorem

If $\mathcal{T}$ is the output of SHORTEST $\mathcal{S}$-EQUIVALENT STRING, *then $|\mathcal{T}| = \mathcal{O}(k + n^2)$.*

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- The shortest cycle has length smaller than $|V|$.
- We obtain total length $\mathcal{O}(\sum_{i=0}^{z}(|V||E| + i|V|)) = \mathcal{O}(z|V||E| + z^2|V|)$.

Total length of the z shortest Eulerian walks on a graph

Theorem

If $||\mathcal{W}_z||$ is the cumulative length of the z shortest Eulerian walks on a graph $G(V, E)$, then $||\mathcal{W}_z|| = \mathcal{O}(z|V|\,|E| + z^2|V|)$ and this bound is asymptotically tight.

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Theorem

*If $\mathcal{T}_z$ is the output of z-*Shortest $\mathcal{S}$-Equivalent String *for n strings of length k each, then $||\mathcal{T}_z|| = \mathcal{O}(zk + zn^2 + z^2 n)$.*

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Definition : semi-Eulerian graph

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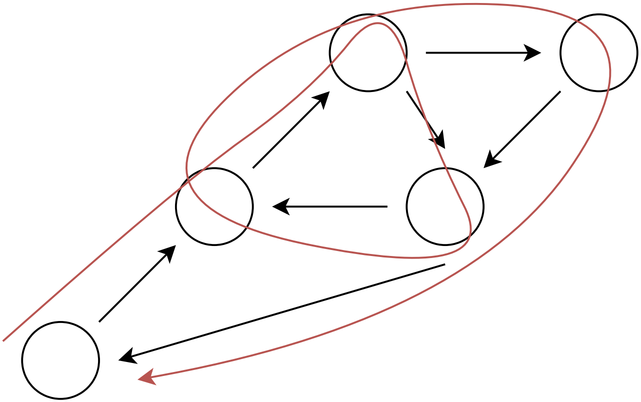
We say that an Eulerian walk is an *Eulerian trail* if it visits every edge *exactly* once.

Definition

We say that a graph G is *semi-Eulerian* if there is an Eulerian trail on G.

Semi-Eulerian extensions



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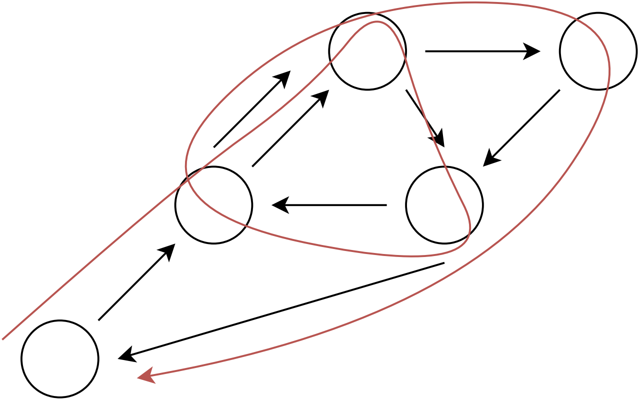

Solving Shortest $\mathcal{S}$-Equivalent String via Eulerian walks

Reminder : To solve Shortest $\mathcal{S}$-Equivalent String and z-Shortest $\mathcal{S}$-Equivalent String, we need to find Eulerian walks in De Bruijn graphs.

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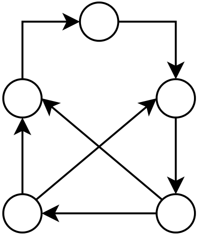
  - Bring back those trails in the original graph to find an Eulerian walk and therefore a $\mathcal{S}$ equivalent string.

CWI

Finding Eulerian walks through flows



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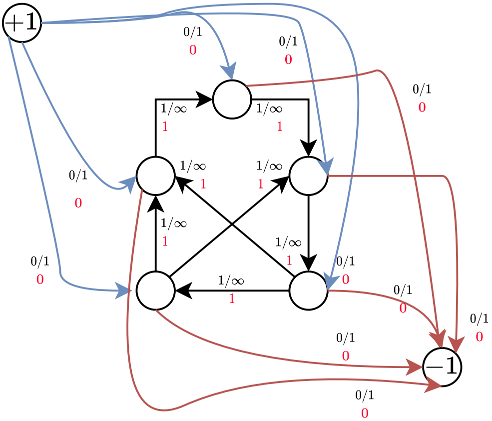

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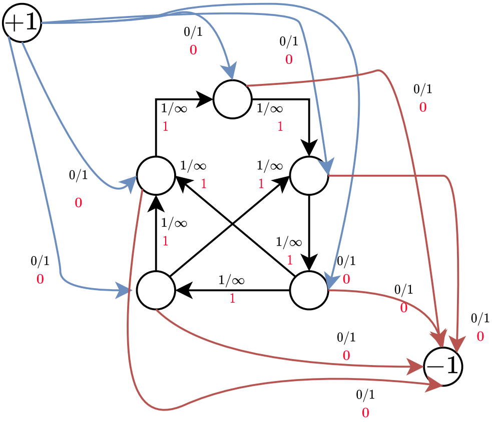
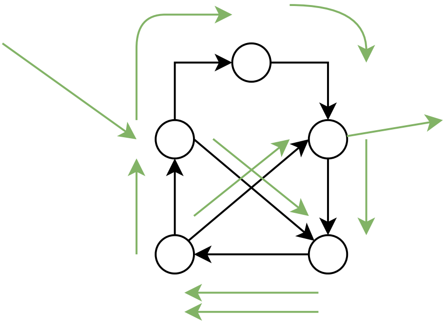


Now we want to model our extension problem with a flow problem.
We extend the graph and create an instance of MINCOSTFLOW.
Each feasible flow on this problem corresponds to a semi-Eulerian extension.

Link between flows on G_{ext} and Eulerian extensions

- We can obtain a minimum cost flow in $\tilde{\mathcal{O}}(|E||V|)$ ([Orlin, 1997],[Tarjan, 1997]).



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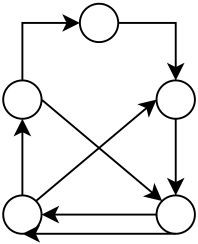

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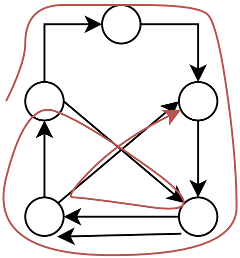

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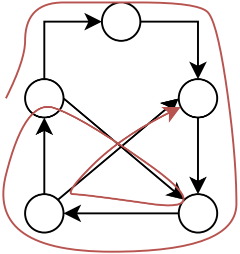

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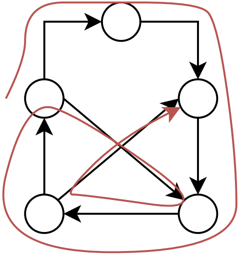
- A minimal cost flow gives us a shortest walk.



Results for Shortest $\mathcal{S}$-Equivalent String

Theorem

The Shortest $\mathcal{S}$-Equivalent String *problem can be solved in* $\tilde{\mathcal{O}}(nk + n^2)$ *time.*

Solving z-Shortest $\mathcal{S}$-Equivalent String

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CWI

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The z-Shortest $\mathcal{S}$-Equivalent String problem can be solved in $\tilde{\mathcal{O}}(nk + zn^2 + ||\mathcal{T}_z||)$ time.

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Theorem

The z-Shortest $\mathcal{S}$-Equivalent String problem can be solved in $\tilde{\mathcal{O}}(nk + zn^2 + ||\mathcal{T}_z||)=\tilde{\mathcal{O}}(nk + zn^2 + zk)$ time.

Thanks for your attention !

This project is part of a project that has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska- Curie grant agreement No 956229.

Co-financed by the Connecting Europe Facility of the European Union

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