# Efficient Enumeration of Regular Languages

Margareta Ackerman    Jeffrey Shallit

School of Computer Science, University of Waterloo

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The $4^{th}$-cross-section is: $0000, 0111, 1011, 1101, 1110$.

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Given words $u = u_1 u_2 \cdots u_n$ and $v = v_1 v_2 \cdots v_m$, $u < v$ according to *radix order* if $n < m$ or if $n = m$, $u \neq v$, and $u_i < v_i$ for the minimal $i$ where $u_i \neq v_i$.

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Enumerate the first 5 words in $L(N)$: $\epsilon, 0, 1, 00, 01$.

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$L(N) = \{w \in \{0,1\}^* \mid |w|_1 \equiv 0 \mod 3\}$
Enumerate the first 10 words in $L(N)$:
$\epsilon, 0, 00, 000, 111, 0000, 0111, 1011, 1101, 1110$.

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- Ex. Verify that an NFA has the property that the least word not accepted is exponential in length (construction by Shallit).
- Test if two NFAs accept the same language.
- If sufficiently many words are generated, then we can verify if two NFAs accept the same language (Conway, 1971).

# Applications

Given an NFA on $s$ states, decide if every word it accepts is a power (a string of the form $x^n, |x| \geq 1, n \geq 2$).

- If every word is a power, then the NFA accepts no more than $7s$ words of each length, and further, if it accepts a non-power, it must accept a non-power of length $< 3s$ (Anderson, Rampersad, Santean, and Shallit.)

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- We get an efficient algorithm by enumerating all the words of length $1, 2, \ldots, 3s - 1$ and testing if each is a power, stopping if the number of words in any cross-section exceeds $7s$.

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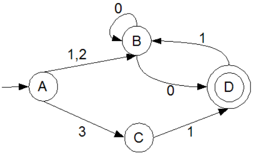
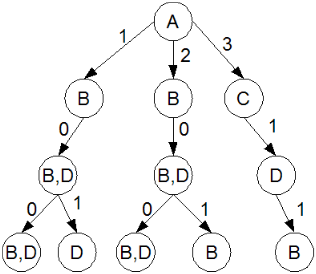
Enumerate the $3^{rd}$-cross-section of the following NFA.




(e) NFA




(f) Illustration of Grail algorithm

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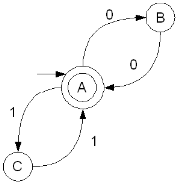

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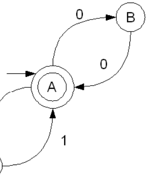


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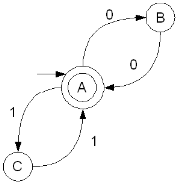


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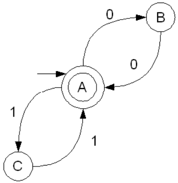


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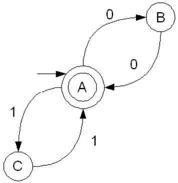


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- Works well in practice for small input sizes.
- We found a few bugs in Grail. Words are not always output in lexicographical order and for some NFAs words are missing from the enumeration.

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- Mäkinen's enumeration algorithm uses $O(s^2 e + \sigma s^2 t)$ operations.
  $e$: number of empty cross-sections. $n$: length of words in cross-section.
  $t$: output size. $s$: number of states in the NFA. $\sigma$: size of the alphabet of the NFA.

# Previous Work

Pál Dömösi gives a cross-section enumeration algorithm, where finding each consecutive word is superexponential in the size of the cross-section (Dömösi, 1998.)

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- We construct an enumeration algorithm through repeated application of the cross-section enumeration algorithm.

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- $M^i_{p,q} = 1$ (using bit-arithmetic) if and only if there is a path from state $p$ to state $q$ of length exactly $i$.

# Lookahead-Matrix Algorithm For Finding the Minimal Word

**Definition**

A state $q$ in an NFA $N$ is *$i$-complete* if there is a path in $N$ of length $i$ starting at $q$ and ending at a final state.

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- Notice that $M^i$ enables us to determine if a given state is $i$-complete.
- To find the minimal word of length $n$:
  - Compute $M, M^2, \ldots, M^n$ using boolean matrix multiplication.
  - Find the set of $(n-1)$-complete states, $S_1$, reachable from the start state on the minimal possible symbol $a_1$.
  - Then find the set of $(n-2)$-complete states, $S_2$, reachable from any state in $S_1$ on the minimal possible symbol $a_2$.
  - Repeat this process for a total of $n$ iterations.
  - Then $a_1 a_2 \cdots a_n$ is the minimal word of length $n$.

# Finding the Minimal Word

## Algorithm

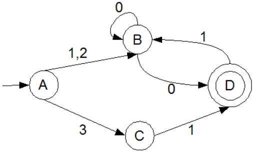
*minWordLM(n,N)*

*INPUT: A nonnegative integer $n$ and an NFA $N$.*
*OUTPUT: The minimal word of length $n$ accepted by $N$. Updates state stack $S$ for a potential subsequent call to minWord or nextWord.*

*Compute* $M, M^2, \ldots, M^n$
$S_0 = \{s_0\}$
*IF* $M^n_{q,f} = 0$ *for all* $f \in F, q \in S_0$
  *return NULL*
$w$ *= empty word*
*FOR* $i \leftarrow 0 \cdots n-1$
  $a_{i+1} = \min(a \in \Sigma \mid \exists u \in S_i, f \in F \textit{ where } M^{n-1-i}_{v,f} = 1 \textit{ for some } v \in \delta(u,a))$
  $w = wa_{i+1}$
  $S_{i+1} = \{v \in \cup_{u \in S_i} \delta(u, a_{i+1}) \mid M^{n-1-i}_{v,f} = 1 \textit{ for some } f \in F\}$
  *IF* $i \neq n-1$
    *push*$(S, S_{i+1})$
*return* $w$

# Finding the Minimal Word

Find the minimal word $w$ of length 4 accepted by the following NFA.

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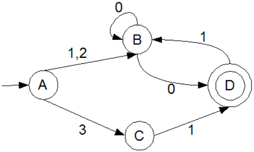


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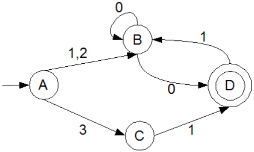


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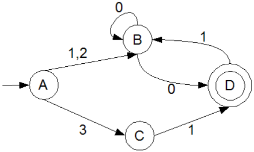


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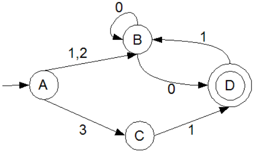


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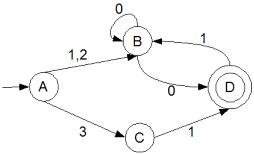


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5. State $D$ is final and reachable from $B$. Set $w = 1000$.

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**Theorem**

*The Lookahead-Matrix algorithm finds the minimal word of length n in $O(s^{2.376}n + \sigma s^2 n)$ time and $O(s^2n)$ space.*

# Finding the Next Word in a Cross-Section

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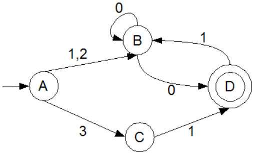
- Continue until the next word is found or the state stack is empty.

# Finding the Next Word

Given the minimal word $w = 1000$, and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the second word $u$ of length 4 accepted by the following NFA.



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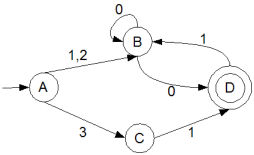


1. Pop $\{B\}$ from $S$. Set $u = 100$, $a = 0$. We cannot reach a final state from $\{B\}$ on a character $> a$.

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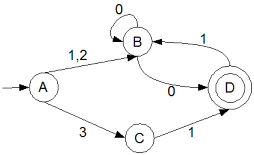


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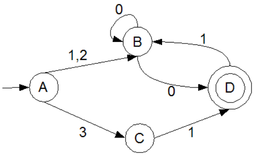


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3. Push $\{B, D\}$ and $\{B\}$ onto $S$. Append 1 to $u$, giving $u = 101$.

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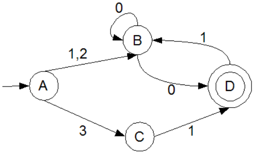


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4. Find the minimal word of length 1 from $\{B\}$ to a final state and append it to $u$.

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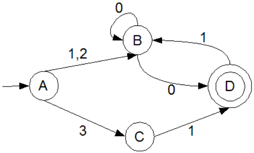


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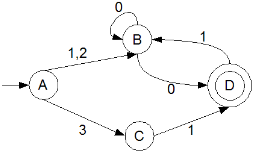
4. Find the minimal word of length 1 from $\{B\}$ to a final state and append it to $u$.
5. We get $u = 1010$ and $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$.

# Finding the Next Word

Given word $u = 1010$ and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the next word $v$ of length 4 accepted by the NFA.



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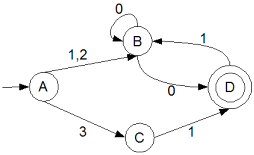


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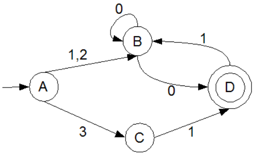


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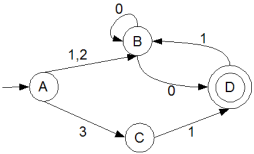


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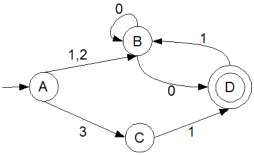


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4. Pop $\{A\}$ from $S$. Now $v = \epsilon$, $a = 1$. State $B$ is a 3-complete state reachable from $\{A\}$ on character $2 > a$.

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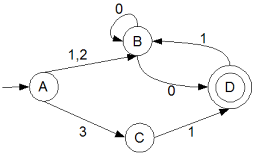


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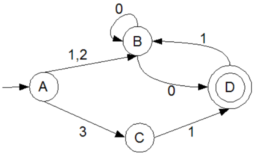


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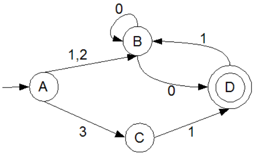


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5. Push $\{A\}$ and $\{B\}$ onto $S$. Append 2 to $v$, giving $v = 2$.
6. Find the minimal word of length 3 from $\{B\}$ to a final state and append it to $v$.
7. Thus, $v = 2000$ and $S = \{\{A\},\{B\},\{B,D\},\{B\}\}$.

# Lookahead-Matrix Cross-Section Enumeration Algorithm

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**Theorem**

*The algorithm* `crossSectionLM` *uses* $O(s^{2.376}n + \sigma s^2 t)$ *operations.*

$t$: output size. $s$: number of states. $\sigma$: alphabet size. $n$: length of words in cross-section.

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**Theorem**

*The algorithm `enumLM` uses $O(s^{2.376}(c + e) + \sigma s^2 t)$ operations.*

$e$: number of empty cross-sections encountered throughout the enumeration.
$c$: number of non-empty cross-sections encountered throughout the enumeration.
$t$: output size. $s$: number of states. $\sigma$: alphabet size.

- The number of consecutive empty cross-sections is at most $s - 1$. Thus, $e < cs < ts$.

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- We give variants of Mäkinen's algorithms which we analyze in the bit-complexity model.

# Mäkinen's Minimal Word Algorithm

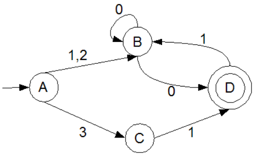
## Algorithm

```
minWordMäkinen(n, N)

INPUT: A positive integer n and an NFA N.
OUTPUT: Table A^min[1···n] for each state A ∈ Q where A^min[i] is the minimal word
of length i starting at state A.

FOR each A ∈ Q
    IF for all a ∈ Σ, δ(A, a) ∩ F = ∅
        A^min[1] = NULL
    ELSE
        A^min[1] = min{a ∈ Σ | δ(A, a) ∩ F ≠ ∅}
FOR i ← 2···n
    FOR each A ∈ Q
        min = NULL
        FOR each B ∈ Q and minimal a ∈ Σ such that B ∈ δ(A, a)
            IF B^min[i − 1] ≠ NULL
                IF aB^min[i − 1] < min OR min = NULL
                    min ← aB^min[i − 1]
        A^min[i] = min
RETURN {A^min | A ∈ Q}
```

Find the minimal word of length 4 in the NFA.




| states/length | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | x | | | |
| B | 0 | | | |
| C | 1 | | | |
| D | x | | | |

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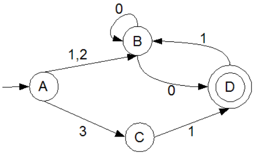


| states/length | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | x | 10 | | |
| B | 0 | 00 | | |
| C | 1 | x | | |
| D | x | 10 | | |

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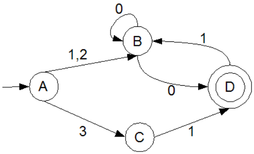
Find the minimal word of length 4 in the NFA.




| states/length | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | x | 10 | 100 | |
| B | 0 | 00 | 000 | |
| C | 1 | x | 110 | |
| D | x | 10 | 100 | |

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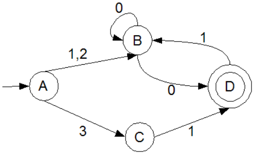


| states/length | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | x | 10 | 100 | 1000 |
| B | 0 | 00 | 000 | 0000 |
| C | 1 | x | 110 | 1100 |
| D | x | 10 | 100 | 1000 |

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# Mäkinen's Minimal Word Algorithm

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- We analyze the algorithm in the *bit-complexity model* and integrate other parameters into our analysis.
- Concatenation of words can be performed in constant time by changing the mode of storage: Instead of storing a word $w$ of length $i$ in $A^{min}[i]$, store the pair $(a, B)$ such that $w = aB^{min}[i-1]$.

| states/length | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | x | (1, B) | (1, B) | (1, B) |
| B | 0 | (0, B) | (0, B) | (0, B) |
| C | 1 | x | x | x |
| D | x | (1, B) | (1, B) | (1, B) |

- With this modification, Mäkinen's algorithm uses $\Theta(sn)$ space to find the minimal word.

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- Since the states of an NFA can form a complete graph, the worst-case running time is $O(s^2n^2)$.
- This worst-case is reached in the figure below.

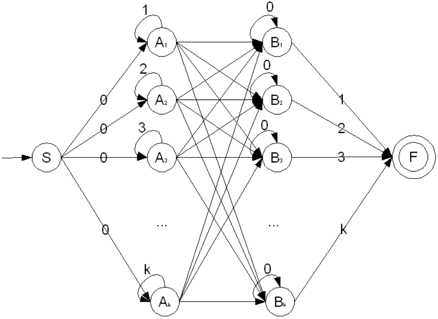


Figure: $\delta(A_i, a_i) = \{B_1, B_2, \ldots, B_k\}$ for all distinct $a_i$.

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- To overcome this constraint, find consecutive words in a cross-section using the same method as in the Lookahead-Matrix algorithm.
- Mäkinen's original cross-section algorithm determines when a cross-section has been fully enumerated by precomputing the maximal word in the cross-section.
- Recall that Lookahead-Matrix determines that a cross-section has been enumerated when the state stack is empty.

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- When Mäkinen's original cross-section termination method is used, we call the algorithm `crossSectionMäkinenI`.

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- When the Lookahead-Matrix cross-section termination is used on Mäkinen's algorithm, we call the algorithm `crossSectionMäkinenII`.

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- When the Lookahead-Matrix cross-section termination is used on Mäkinen's algorithm, we call the algorithm `crossSectionMäkinenII`.
- Both `crossSectionMäkinenI` and `crossSectionMäkinenII` use $O(s^2n^2 + \sigma s^2 t)$ operations.
  $t$: output size. $s$: number of states. $\sigma$: alphabet size.

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- When the Lookahead-Matrix cross-section termination is used on Mäkinen's algorithm, we call the algorithm `crossSectionMäkinenII`.
- Both `crossSectionMäkinenI` and `crossSectionMäkinenII` use $O(s^2n^2 + \sigma s^2 t)$ operations.
  $t$: output size. $s$: number of states. $\sigma$: alphabet size.
- In practice, `crossSectionMäkinenI` and `crossSectionMäkinenII` perform differently.

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- The algorithm `enumMäkinenI` consists of repeated calls to `crossSectionMäkinenI`.
- Similarly, `enumMäkinenII` consists of repeated calls to `crossSectionMäkinenII`.
- Both algorithms use $O(\sigma s^2 t + s^2 e)$ operations.
  $e$: number of empty cross-sections. $t$: output size.
  $s$: number of states. $\sigma$: alphabet size.

# Complexity Summary

| | Cross-Section | Enum |
|---|---|---|
| Lookahead-Matrix | $O(s^{2.376}n + \sigma s^2 t)$ | $O(s^{2.376}(c+e) + \sigma s^2 t)$ |
| Mäkinen | $O(s^2 n^2 + \sigma s^2 t)$ | $O(s^2 e + \sigma s^2 t)$ |
| Grail | $O(s^2 \sigma^{n+1})$ | $O(s^2 \sigma^{k+1})$ |

$e$: number of empty cross-sections. $c$: number of non-empty cross-sections.
$t$: output size. $s$: number of states. $\sigma$: alphabet size.
$n$: length of words in cross-section. $k$: length of words in last cross-section examined.

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- For comparison and correctness testing, we implemented a naive algorithm that generates all words over $\Sigma^*$ and checks which are accepted by the input NFA.

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- The Grail algorithms tend to perform well on small input size.
- The Grail algorithms outperform the other enumeration algorithms on $1^*$.
- Naive and Grail algorithms are significantly slower than the Lookahead-Matrix algorithm and Mäkinen's algorithms on most NFAs.

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- On NFAs where Mäkinen's cross-section enumeration algorithms are quadratic in $n$, `crossSectionLM` performs significantly better than Mäkinen's cross-section algorithms (at times over 50 times faster).
- On average, the Lookahead-Matrix algorithms perform almost as well as the MäkinenII algorithms and better than the MäkinenI algorithms.
- On average, MäkinenII performs best.

# Conclusions

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- The algorithm `crossSectionLM` has the best worst-case complexity and the best worst-case running time in practice.
- The MäkinenII algorithms for both enumeration problems have the best average-case running times in practice.

# Future Work

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- Find heuristics to further improve the running time of the algorithms in practice. For instance, check the density of the language and select algorithm based on the density.
- Prove lower bounds for the enumeration and cross-section enumeration problems.