Average Value and Variance of Pattern Statistics in Rational Models

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Pattern Statistics

finite alphabet A

pattern $a \in A$
$w \in A^+$, $|w| = m$
$R \subseteq A^*$, finite, regular, ...

random text $x \in A^+$, $|x| = n$
stochastic model (Bernoullian, Markovian, ...)

$O_n = \sharp\{$ occurrences of patterns in $x\}$ (positions)

$$O_n \in \{0, 1, \dots, n\}$$

Goals: - asymptotics expressions for $E(O_n)$, $var(O_n)$, ... ;
- limit distributions, $Pr\{O_n \leq z\} \sim \cdots$
- local limit properties, $Pr\{O_n = k\} \sim \cdots$

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Special case: pattern $w \in A^m$

$$O_n(w) = |x|_w \ , \quad O_n(w) \in \{0, 1, \ldots, n - m + 1\}$$

Example: $w = acba$

$x = cbacbacbababcacbaccacbabaac \cdots$

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Motivations

- Code theory

  [Guibas-Odlyzko 78, 81]
- Statistical analysis of DNA sequences

  [Prum - Rodolph - de Turckheim 95], [Gelfand 95] ...
- Approximation algorithms for string-matching

  [Szpankowski 96, 98, 05]
- Max coefficients in rational formal series

  [Reutenauer,...,Wich]
- Additive functions of numeration systems on regular languages

  [Grabner-Rigo]

Previous works

$$E(O_n), var(O_n), Pr\{O_n \leq z\} \sim \cdots, Pr\{O_n = k\} \sim \cdots$$

1) pattern: $w \in A^m$
text: $x \in A^n$ Markovian sequence, (primitive, stationary)
[Regnier-Spankovski 98], [Jacket-Spankovski 05]

2) pattern: $R \subseteq A^*$ regular
text: $x \in A^n$ Markovian sequence, (primitive)
[Flajolet-Nicodeme-Salvi 02] ...

3) pattern: $a \in A$
text: $x \in A^n$ rational model, $r \in \mathbb{R}_+^{rat}\langle\langle A \rangle\rangle$
(primitive, bicomponent, multicomponent)
[BCGL03, 06], [DGL04], [GL06]

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Our contribution

$E(O_n)$, $var(O_n)$

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text: $x \in A^n$ rational model, $r \in \mathbb{R}_+^{\text{rat}}\langle\langle A \rangle\rangle$
primitive condition

$$E(O_n) = \beta n + \delta + \mathrm{O}(\varepsilon^n)$$
$$var(O_n) = \gamma n + \mathrm{O}(1)$$

$\beta, \ \delta, \ \gamma \ \longleftrightarrow$ linear representation of r

Markovian sequences $\Longleftrightarrow$ Uniform dist. on $L \cap A^n$ (L regular)

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Bernoullian models

$p : A \longrightarrow [0,1]$ $\sum_{a \in A} p(a) = 1$

random text $x = x_1 x_2 \cdots x_n \in A^+$ $(x_i \in A)$:
- $\Pr\{x_i = a\} = p(a)$
- $\{x_i = a\}$ and $\{x_j = b\}$ are independent $(i \neq j)$.

$$\Pr\{x = a_1 a_2 \cdots a_n\} = \prod_{i=1}^{n} p(a_i)$$

- gen. funct. of $\{\sharp(A^n \backslash A^* R A^*)\}$ $(\Pr(O_n(R) = 0))$

  [Guibas-Odlyzko 81]

- limit distribution of $O_n(R)$ with partially hidden patterns $R \subseteq A^+$ (special regular)

  [Flajolet et al. 06]

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Markovian sequences of order m

$\pi : A^m \longrightarrow [0,1]$ (initial) prob. dist.

$p : A^m \times A \longrightarrow [0,1]$ $\sum_{a \in A} p(y,a) = 1 \quad (\forall y \in A^m)$

$$x_1 x_2 \cdots \underbrace{x_{i-m} \cdots x_{i-1}}_{y} \underbrace{x_i}_{a} \cdots x_n$$

$$\Pr\{x_i = a \mid x_{i-m} \cdots x_{i-1} = y\} = p(y,a) \quad (\forall i > m)$$

$$\Pr\{x = a_1 a_2 \cdots a_n\} = \pi(a_1^m) \prod_{i=m+1}^{n} p(a_{i-m}^{i-1}, a_i)$$

- under primitive and stationary hypothesis :

  $E(O_n(w)) \sim \ldots$, $var(O_n(w)) \sim \ldots$,
  Gaussian limit distribution of $O_n(w)$

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- similar results for $O_n(R)$ with regular R

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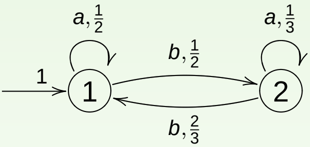
Markovian models

Markov chain over $Q = \{1, \ldots, k\}$

$+$ A-labels on transitions (deterministic choice)



Definition: $\mathcal{M} = (\pi, \mu)$

- $\pi \in [0,1]^k$ — stochastic vector
- $\mu : A \longrightarrow [0,1]^{k \times k}$ — $\forall a, i \ \exists$ at most one $j : \mu(a)_{ij} \neq 0$

  $M = \sum_{a \in A} \mu(a)$ stochastic matrix

$$\Pr_{\mathcal{M}}(x_1 x_2 \cdots x_n) = \pi' \mu(x_1) \cdots \mu(x_n) \underline{1}$$

$$\Pr_{\mathcal{M}} \in \mathbb{R}_+^{rat}\langle\langle A \rangle\rangle$$

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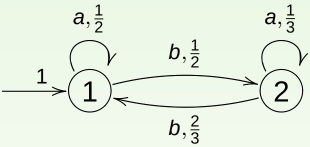

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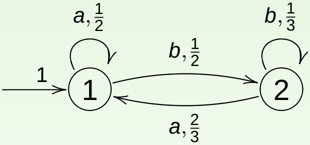
Properties :

1. $\forall m$ Markovian sequences (m) $\subset$ Markovian models



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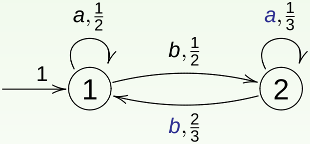

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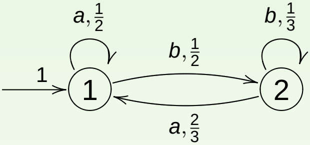

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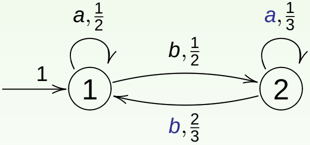

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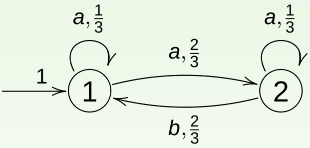

Definition: $\mathcal{S} = (\pi, \mu)$

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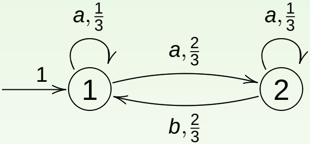

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Properties :

- $\mathcal{S} = (\pi, \mu) \equiv$ Stochastic Sequential Machines (over 1^*)

  [Paz 69]

- $\Pr_{\mathcal{S}} \equiv$ Probability Measure over A^*

  [Hansel-Perrin 90]

- Strict inclusion: Markovian Models $\subsetneq$ Sequential Models

$\Pr_{\mathcal{S}}(a^n) = (n+1)3^n$

The iteration lemma does not hold.

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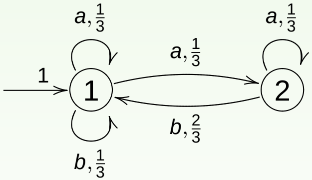

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Rational models

Idea : $L \subseteq A^*$ regular
generate $x \in L \cap A^n$ under uniform distribution
($\exists$ linear time algorithms [Flajolet et al.], [Denise])

Definition $r : A^* \longrightarrow \mathbb{R}_+$ s.t.

- $r \in \mathbb{R}_+^{rat}\langle\langle A \rangle\rangle \equiv (\xi, \mu, \eta) : \begin{cases} \xi, \eta \in \mathbb{R}_+^k & \text{initial and final array} \\ \mu : A^* \to \mathbb{R}_+^{k \times k} & \text{monoid morphism} \\ r(x) = \xi' \mu(x) \eta & \end{cases}$

- $\Pr_r(x) = \dfrac{r(x)}{\sum_{y \in A^n} r(y)} \qquad \forall x \in A^n$

Observations

- $r = \chi_L \implies \Pr_r \equiv$ uniform distribution over $L \cap A^n$
- $\Pr_r =$ Hadamard division of rational f.s.

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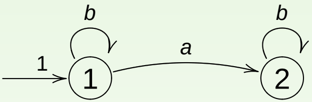

$$\Pr_r(x) = \begin{cases} \frac{1}{|x|+1} & \text{if } |x|_a \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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- Reduction to symbol occurrences:

$$\forall r \in \mathbb{R}_+^{rat}\langle\langle A \rangle\rangle \quad \forall R \subseteq A^* \text{ regular} \qquad \exists s \in \mathbb{R}_+^{rat}\langle\langle a, b \rangle\rangle \ :$$

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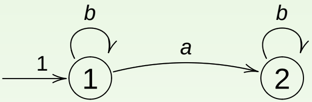

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Assumptions

Rational model

- linear representation (ξ, μ, η)

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Ergodic condition

- $M = \sum_{a \in A} \mu(a)$ primitive (irreducible and aperiodic)

$\exists\ \lambda$ maximum eigenvalue (Perron-Frobenius)

$$\Longrightarrow \quad M^n = \lambda^n (uv' + C(n))$$

$$\begin{cases} v, u & \text{l. and r. eigenvectors of } M\ (\leftrightarrow \lambda) : v'u = 1 \\ C(n) = O(\varepsilon^n), & (|\varepsilon| < 1) \\ C = \sum_0^\infty C(n) & \text{fundamental matrix} \end{cases}$$

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Average value

pattern: $w \in A^m$

$$E(O_n) \; = \; \beta \, (n - m + 1) \; + \; a \; + \; b + \; O(\varepsilon^n) \; , \qquad (|\varepsilon| < 1)$$

$$\beta = \frac{v'\mu(w)u}{\lambda^m} \qquad \approx \; \text{prob of } w \text{ in the middle of } x$$

$$a = \frac{\xi' C \mu(w) u}{\lambda^m \xi' u}, \qquad b = \frac{v'\mu(w) C \eta}{\lambda^m v' \eta}$$

Special cases

- $m = 1$ [BCGL02]
- sequential models

$$E(O_n) \; = \; v'\mu(w)\underline{1} \, (n - m + 1) \; + \; \underbrace{\xi' C \mu(w) \underline{1}}_{= 0 \text{ in the stationary case } (\xi = v)} + O(\varepsilon^n)$$

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Variance

Under the same assumptions

$$Var(O_n) = \gamma n + c + O(\varepsilon^n), \qquad (|\varepsilon| < 1)$$

$$\gamma = \beta - (2m-1)\beta^2 + 2\frac{v'\mu(w)\left[C\mu(w) + P(w)\right]u}{\lambda^{2m}}$$

$$P(w) = \sum_{k \in S} \lambda^k \mu(w_{k+1}^m), \quad S = \{k \in \{1, 2, \ldots, m-1\} \mid w_1^k = w_{m-k+1}^m\}$$

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- simpler expression in the sequential models

$$\gamma = \beta - (2m-1)\beta^2 + 2v'\mu(w)\left[C\mu(w) + \sum_{k\in S}\mu(w_{k+1}^m)\right]\underline{1}$$

- Markovian sequences [Regnier-Szpankowski 98]

Conclusions

We have seen:

- Moments of (word) pattern statistics for more general stochastic models
- The known evaluations for the Markovian texts hold in the sequential models
- the stationary hypothesis affects the constant term of the moments

Question: similar extentions hold for the limit distributions?

THANK YOU FOR ATTENTION

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