Synchronizing Automata Preserving a Chain of Partial Orders

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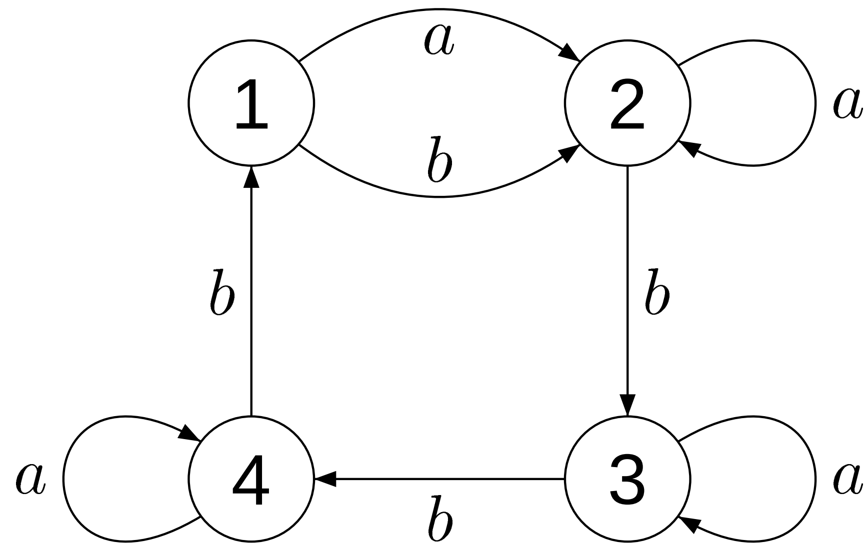
Any word w with this property is said to be a *reset word* for the automaton.

Synchronizing automata



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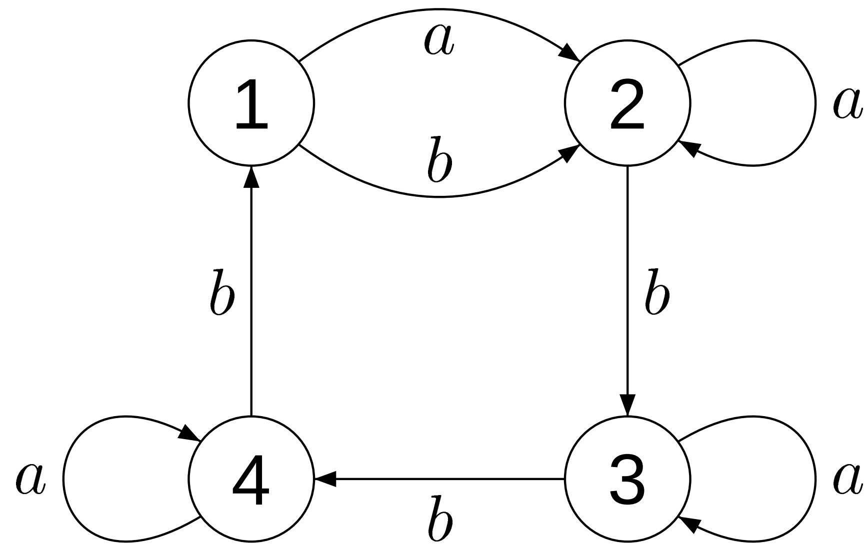

A reset sequence of actions is $abbbabbba$. Applying it at any state brings the automaton to the state 2.

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- The idea of synchronization is pretty natural and of obvious importance: we aim to restore control over a device whose current state is not known.
- Think of a satellite which loops around the Moon and cannot be controlled from the Earth while “behind” the Moon (Černý’s original motivation).

Engineering Applications

In the 80s, the notion was reinvented by engineers working in *robotics* or, more precisely, *robotic manipulation* which deals with part handling problems in industrial automation such as part feeding, fixturing, loading, assembly and packing (and which is therefore of utmost and direct practical importance).

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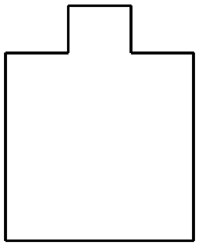

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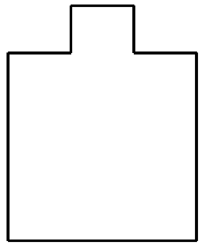

Such parts arrive at manufacturing sites in boxes and they need to be sorted and oriented before assembly.

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Suppose that prior the assembly the part should take the “bump-left” orientation (the second one on the picture). Thus, one has to construct an orienter which action will put the part in the prescribed position independently of its initial orientation.

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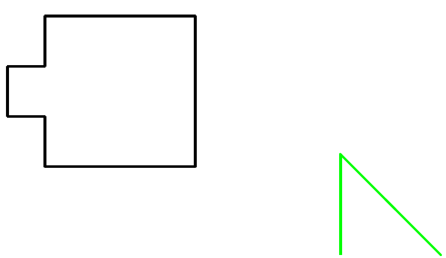
A high obstacle is high enough so that any part on the belt encounters this obstacle by its rightmost low angle.



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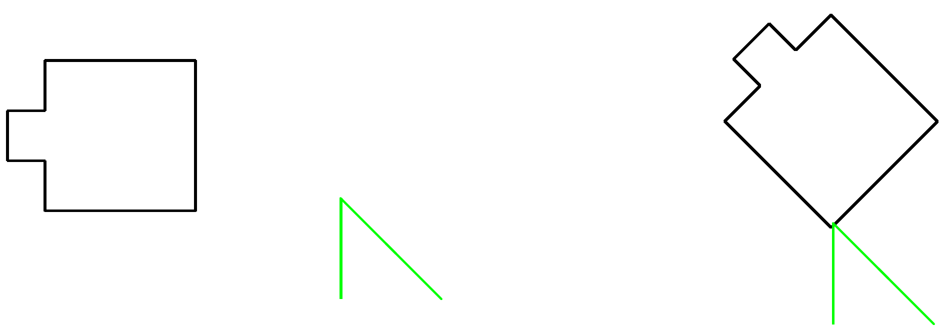

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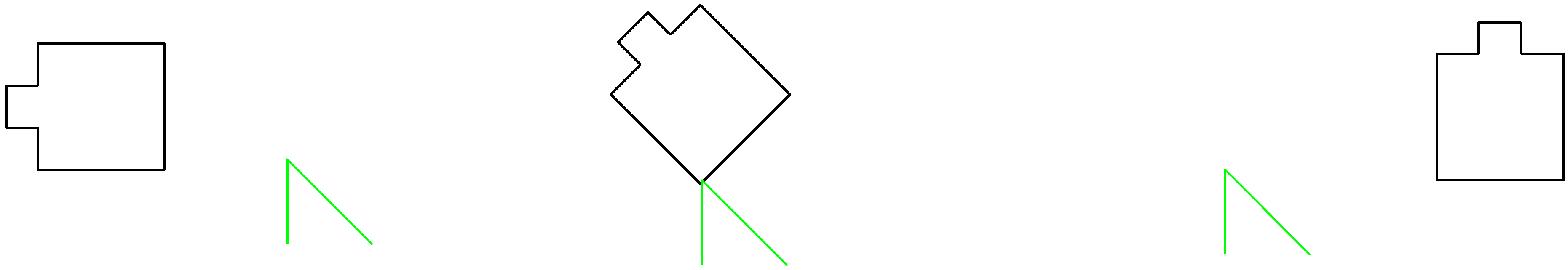

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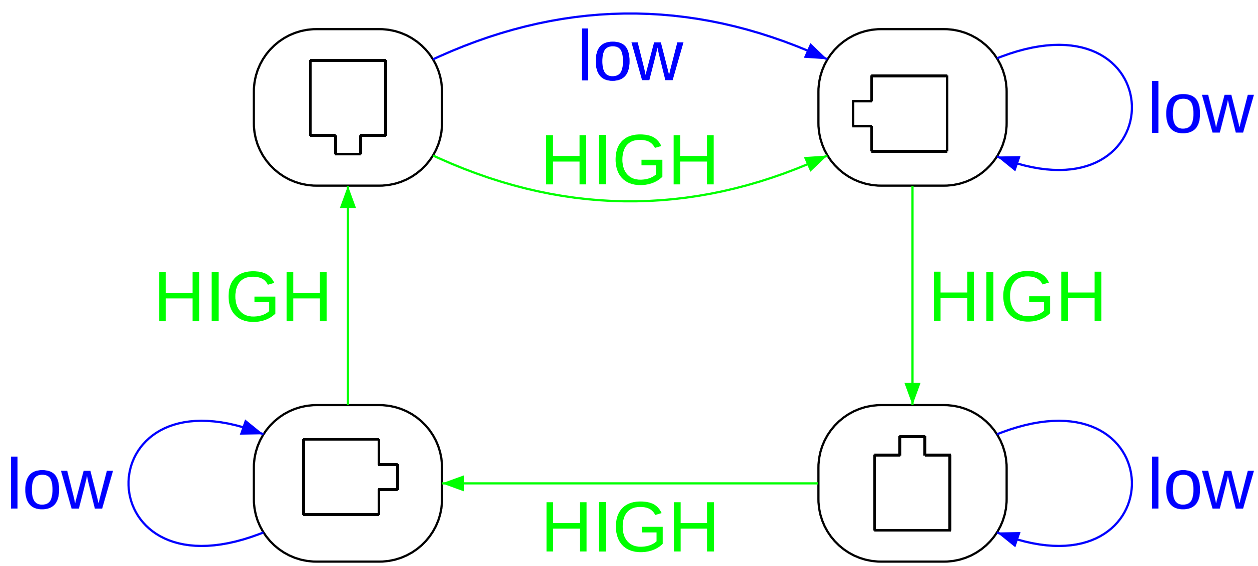
The following schema summarizes how the obstacles effect the orientation of the part in question:



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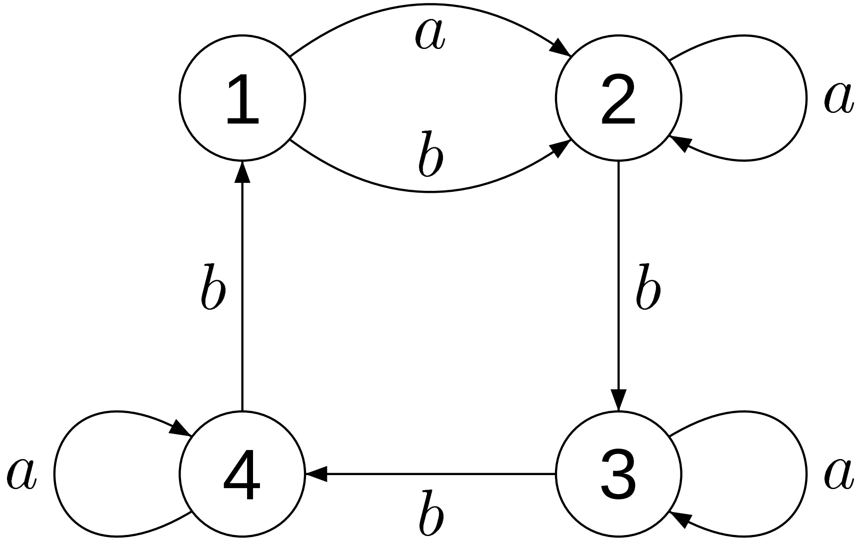

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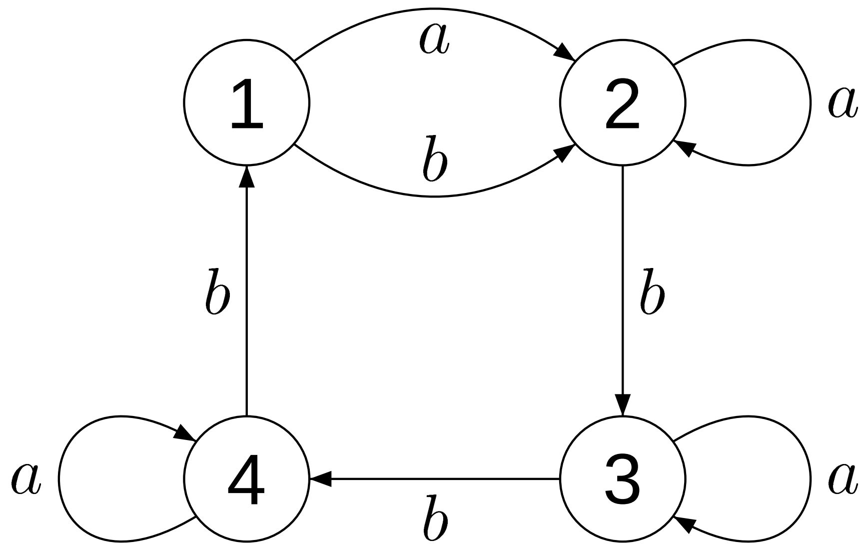

– this was our example of a synchronizing automaton, and we saw that $abbbabbba$ is a reset sequence of actions. Hence the series of obstacles

low-HIGH-HIGH-HIGH-low-HIGH-HIGH-HIGH-low

yields the desired sensorless orienter.

Further Applications

In *DNA-computing*, there is a fast progressing work by Ehud Shapiro's group on "*soup of automata*" (Programmable and autonomous computing machine made of biomolecules, Nature 414, no.1 (November 22, 2001) 430–434; DNA molecule provides a computing machine with both data and fuel, Proc. National Acad. Sci. USA 100 (2003) 2191–2196, etc).

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The simply looking conjecture is still open in general!!

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Given a DFA $\mathscr{A}$ and a positive integer ℓ, the problem whether or not $\mathscr{A}$ has a reset word of length $\leq \ell$ is NP-complete (D. Eppstein, 1990; P. Goralčik and V. Koubek, 1995; A. Salomaa, 2003).

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Given a DFA $\mathscr{A}$ and a positive integer ℓ, the problem whether or not the shortest reset word for $\mathscr{A}$ has length ℓ is co-NP-hard (W. Samotij, 2007).

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- Synchronization issues remain difficult when restricted to $\mathbf{Ap}$.

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The gap between the upper and the lower bounds is rather drastic.

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An idea: consider certain properties that guarantee aperiodicity and are easier to check.

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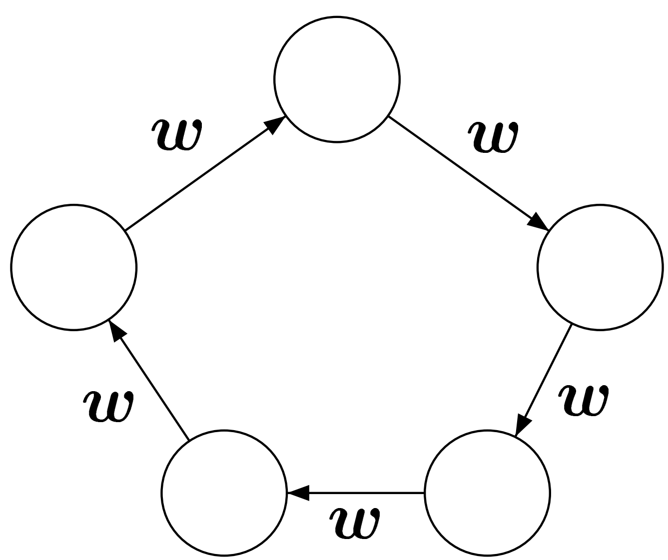

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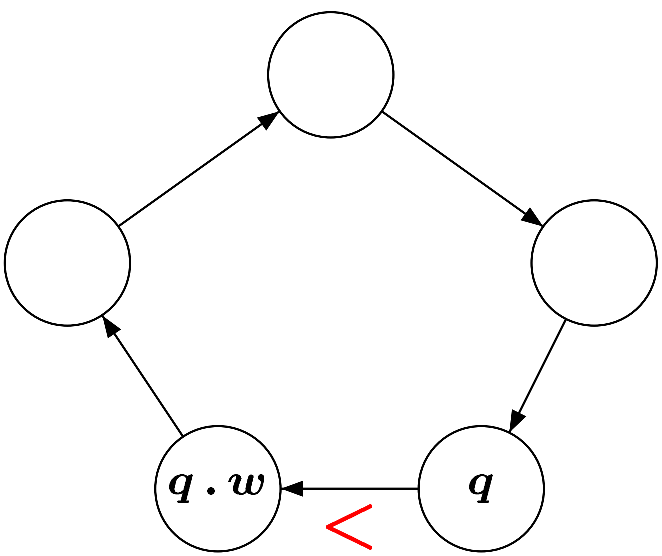

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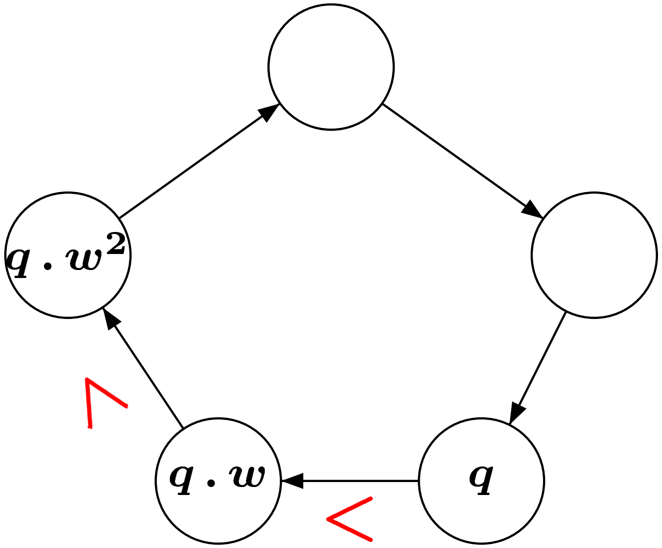

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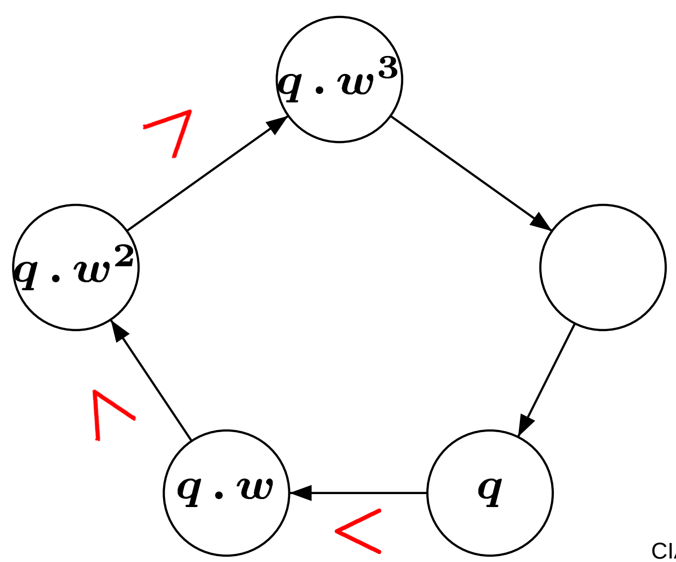

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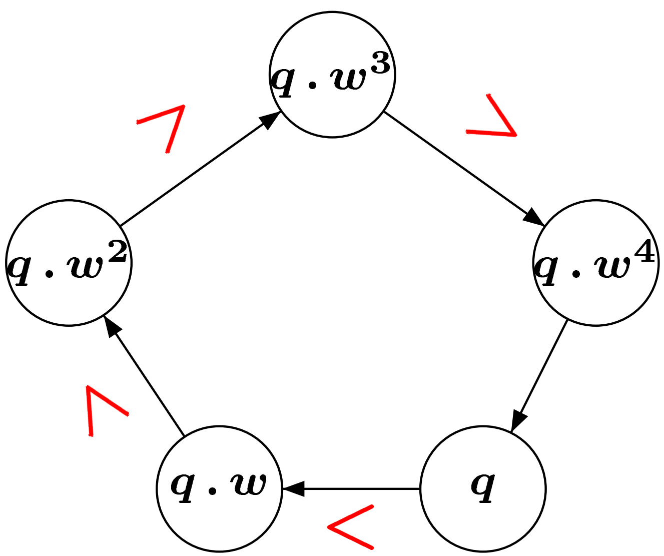

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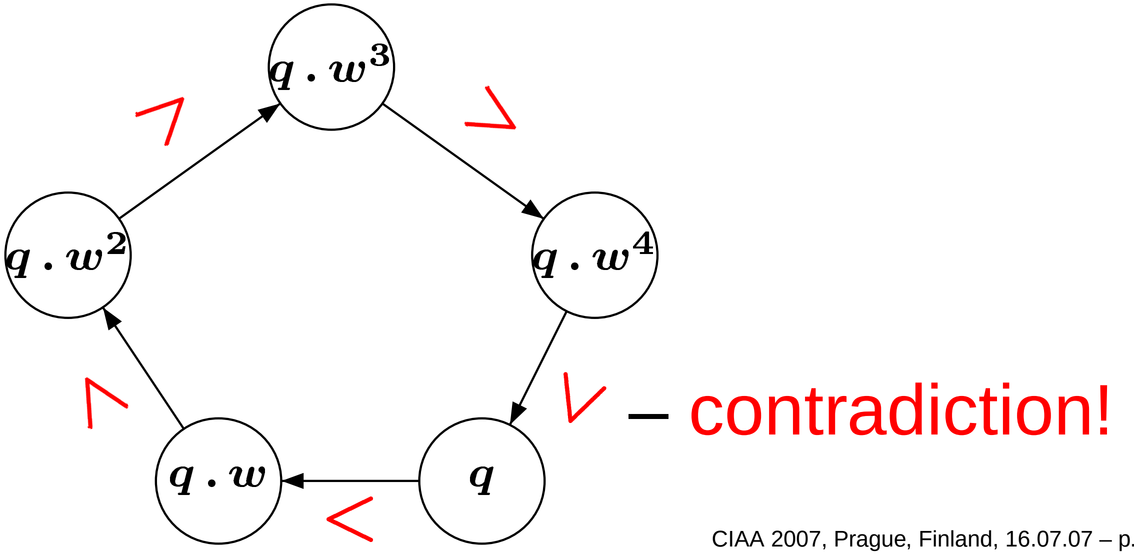

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A much more difficult case of so-called 0-monotonic automata was analyzed by Ananichev in 2005. A DFA $\mathscr{A} = \langle \boldsymbol{Q}, \boldsymbol{\Sigma}, \boldsymbol{\delta} \rangle$ is 0-*monotonic* if it has a unique *sink* $\boldsymbol{s} \in \boldsymbol{Q}$ and $\boldsymbol{Q} \setminus \{\boldsymbol{s}\}$ admits a linear order $\leq$ preserved by the restrictions of the transformations $\boldsymbol{\delta}(\sqcup, \boldsymbol{a})$ to $\boldsymbol{Q} \setminus \{\boldsymbol{s}\}$.

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A much more difficult case of so-called 0-monotonic automata was analyzed by Ananichev in 2005. A DFA $\mathscr{A} = \langle \boldsymbol{Q}, \Sigma, \boldsymbol{\delta} \rangle$ is 0-*monotonic* if it has a unique *sink* $\boldsymbol{s} \in \boldsymbol{Q}$ and $\boldsymbol{Q} \setminus \{s\}$ admits a linear order $\leq$ preserved by the restrictions of the transformations $\boldsymbol{\delta}(\sqcup, \boldsymbol{a})$ to $\boldsymbol{Q} \setminus \{s\}$. Clearly, 0-monotonic automata are in a 1-1 correspondence with incomplete monotonic automata.

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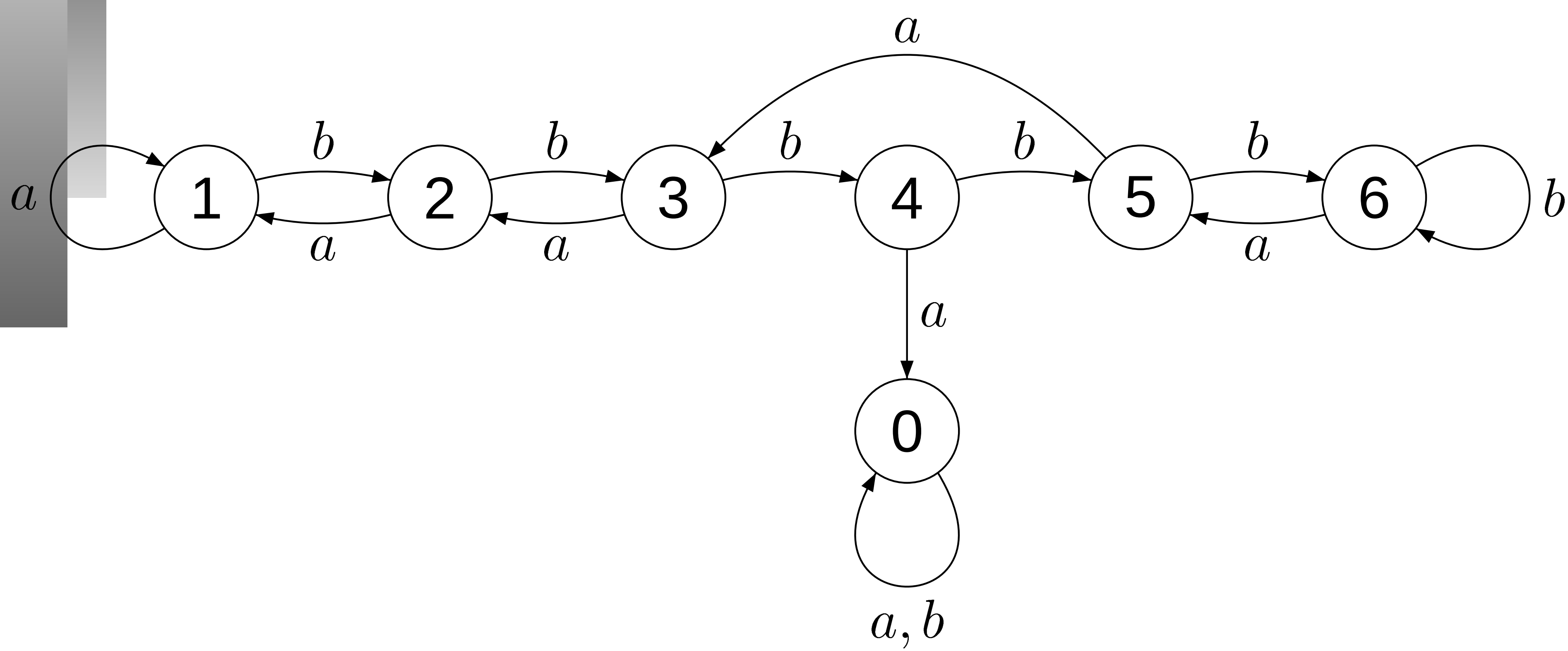
Monotonicity



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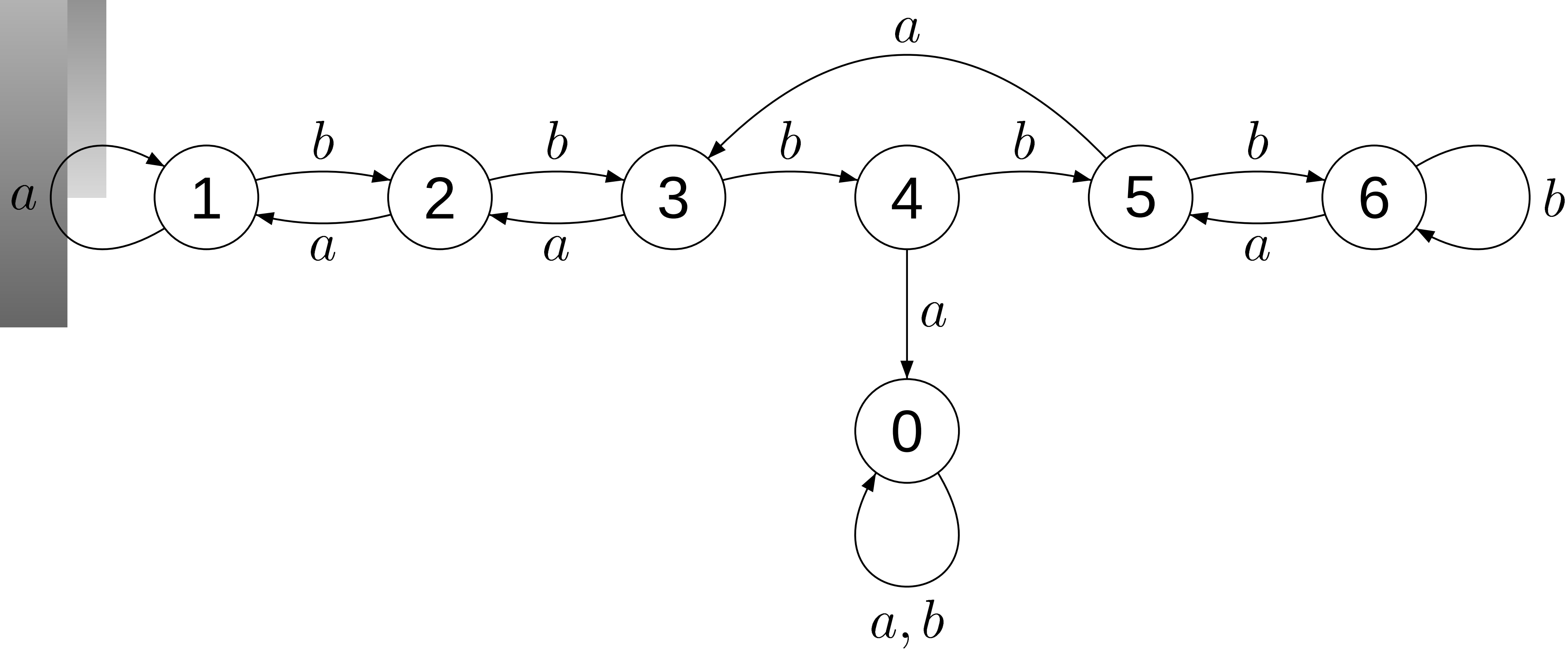

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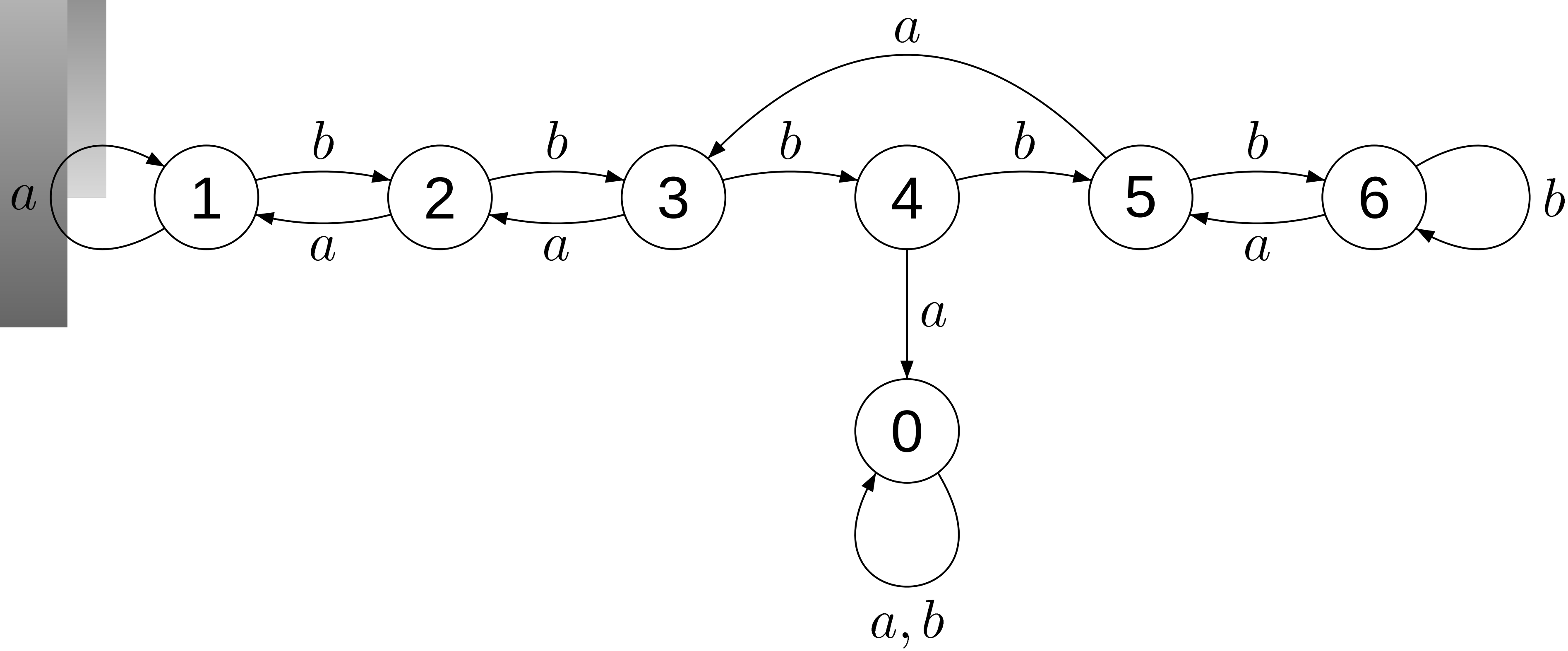

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It has 7 states and its shortest reset word is a^4b^3a of length $7 + \left\lfloor \frac{7}{2} \right\rfloor - 2 = 8$.

Generalized Monotonicity

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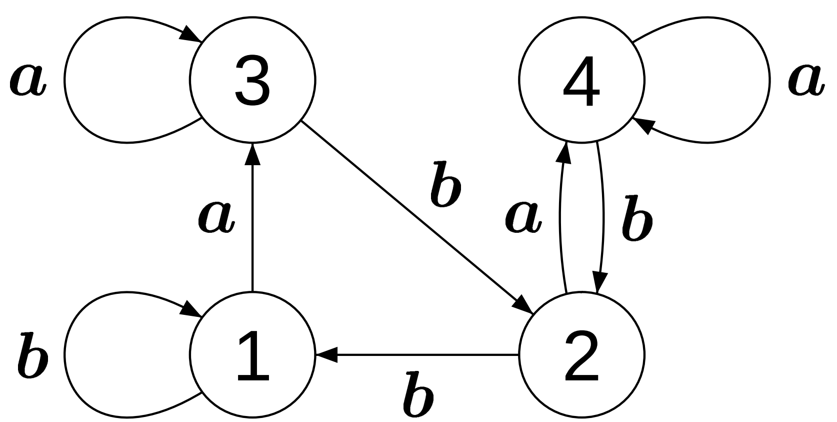

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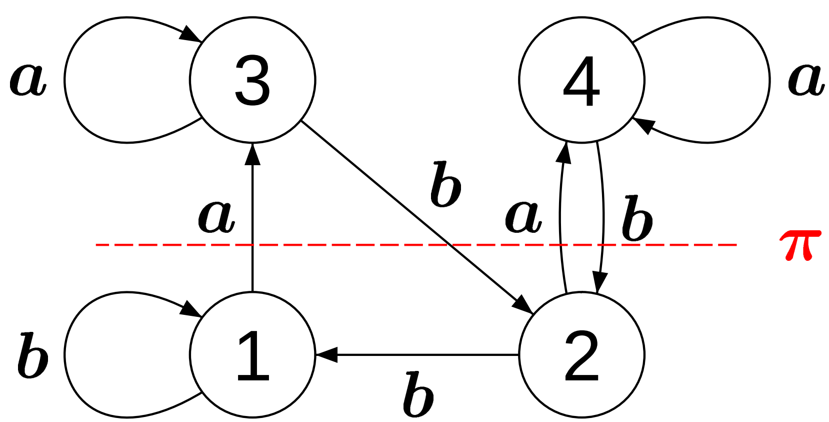

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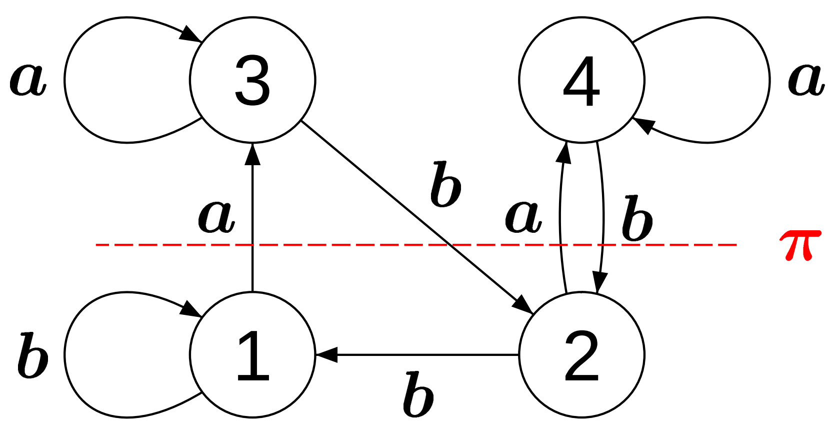

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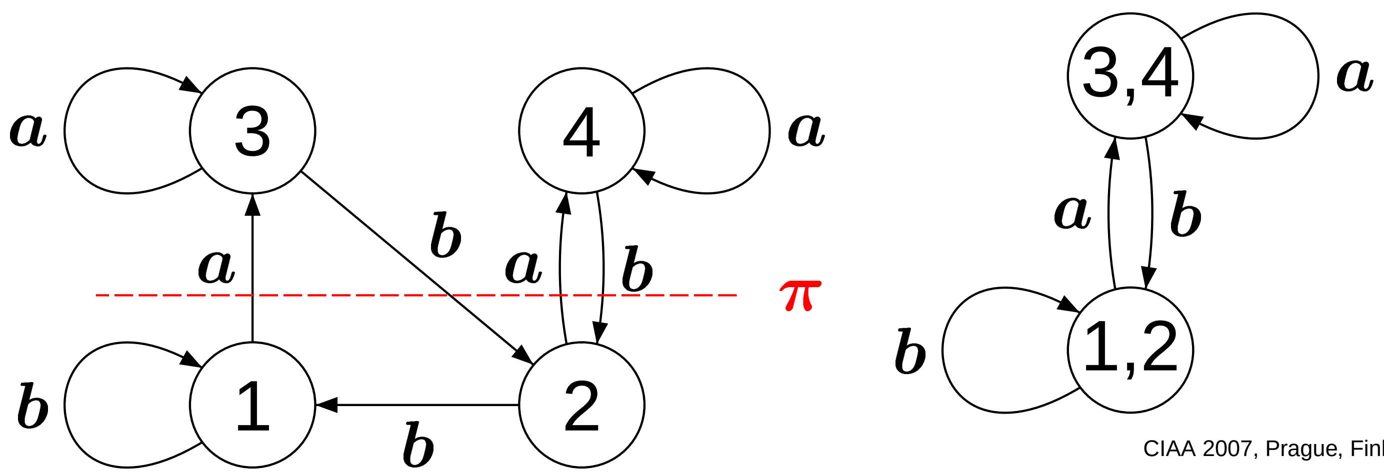

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The automaton in the previous example is a generalized monotonic automaton of level 2.

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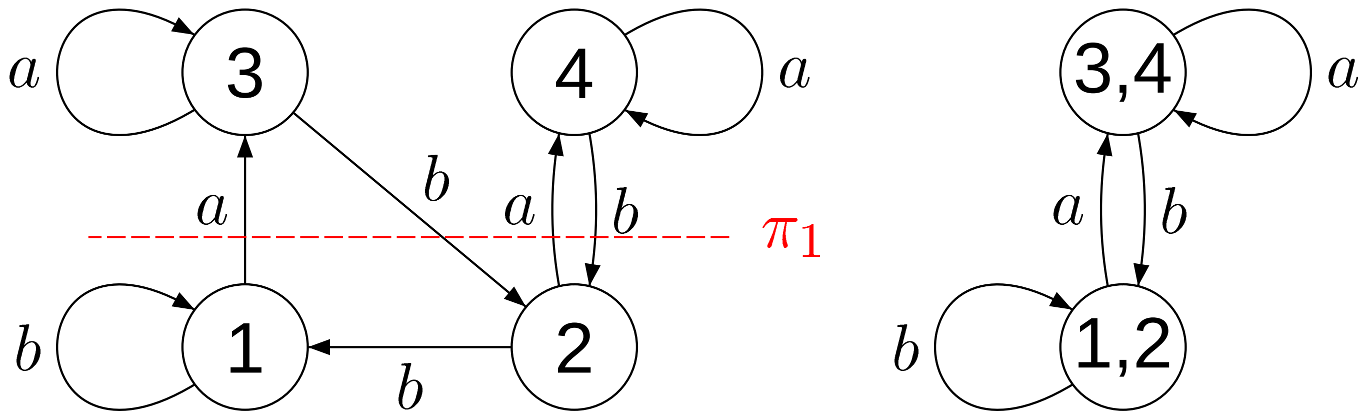

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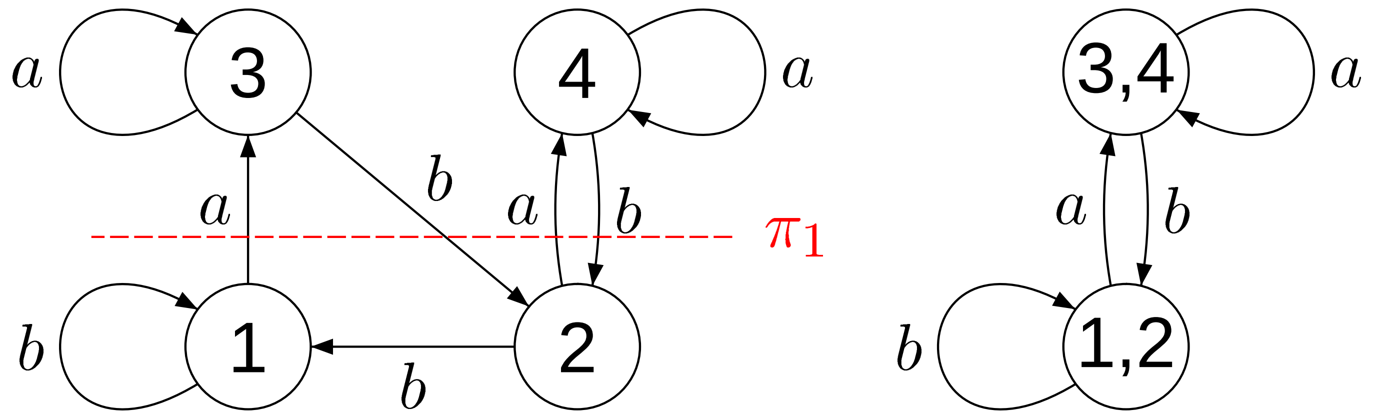

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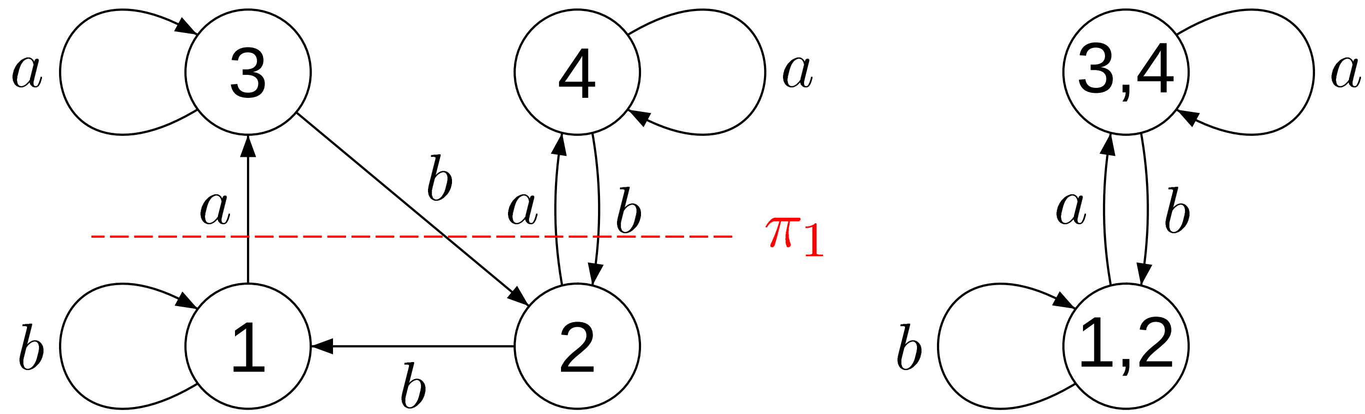

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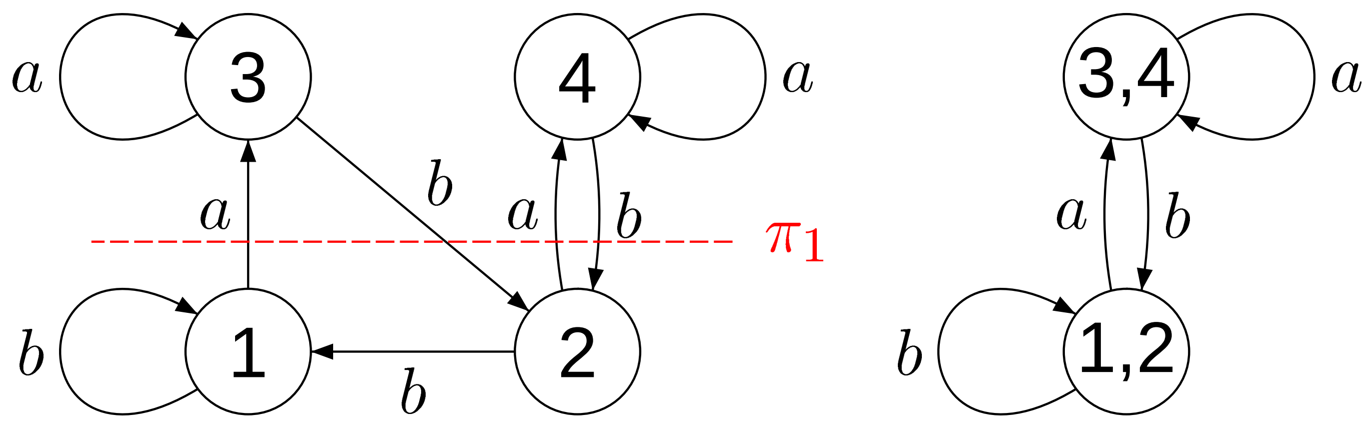

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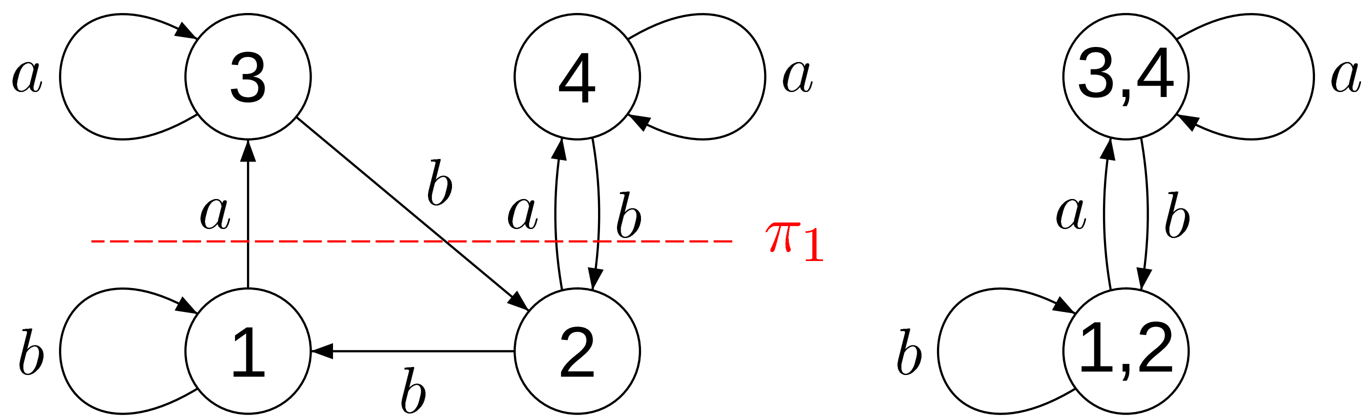

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It can be shown that the automaton is not monotonic. Moreover, it cannot be emulated by any monotonic automaton.

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However, generalized monotonic automata are not representative for the class $\mathbf{Ap}$ from the synchronization point of view: Ananichev and $\sim$ (2005) proved that every generalized monotonic synchronizing automaton with n states has a reset word of length $\leq n - 1$.

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Thus, we just dropped the restriction that the order ρ_i/π_{i-1} is linear on each π_i/π_{i-1}-class.

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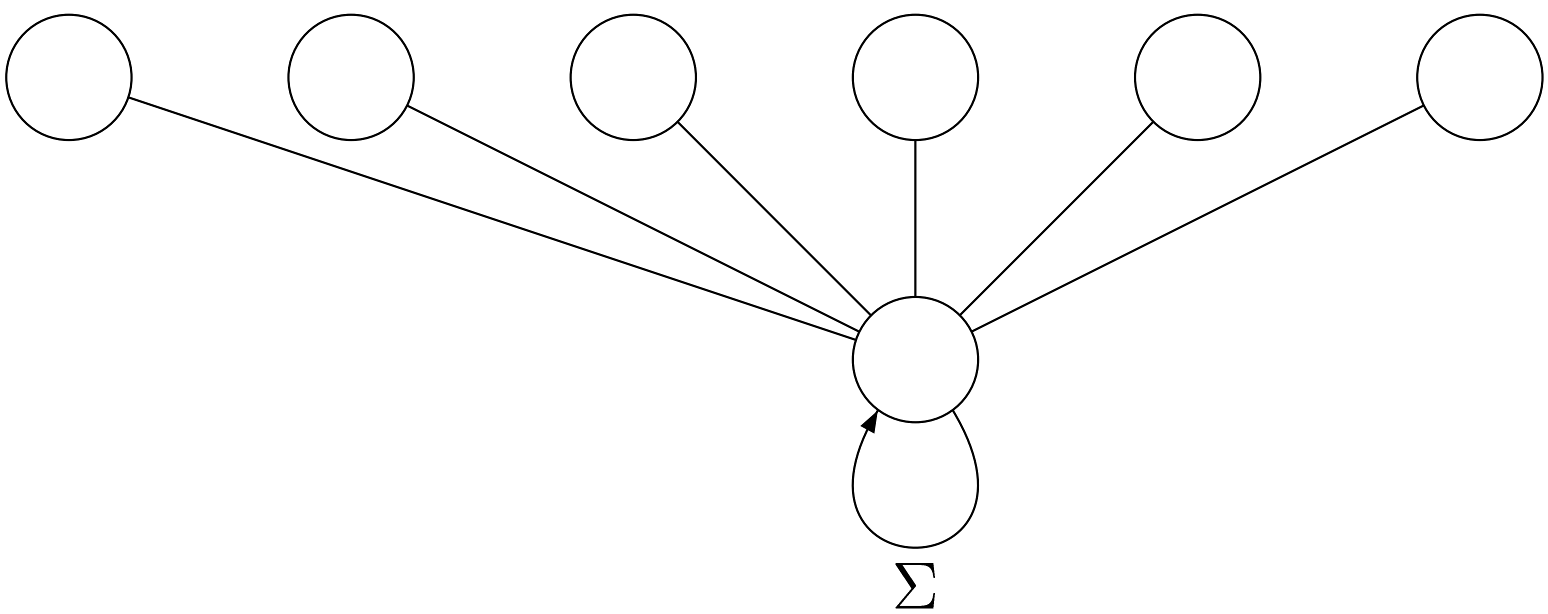

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- Every weakly monotonic automaton with a strongly connected underlying digraph is synchronizing. (A non-trivial generalization of the corresponding result for aperiodic automata.)
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