Prague Stringology Conference 2024

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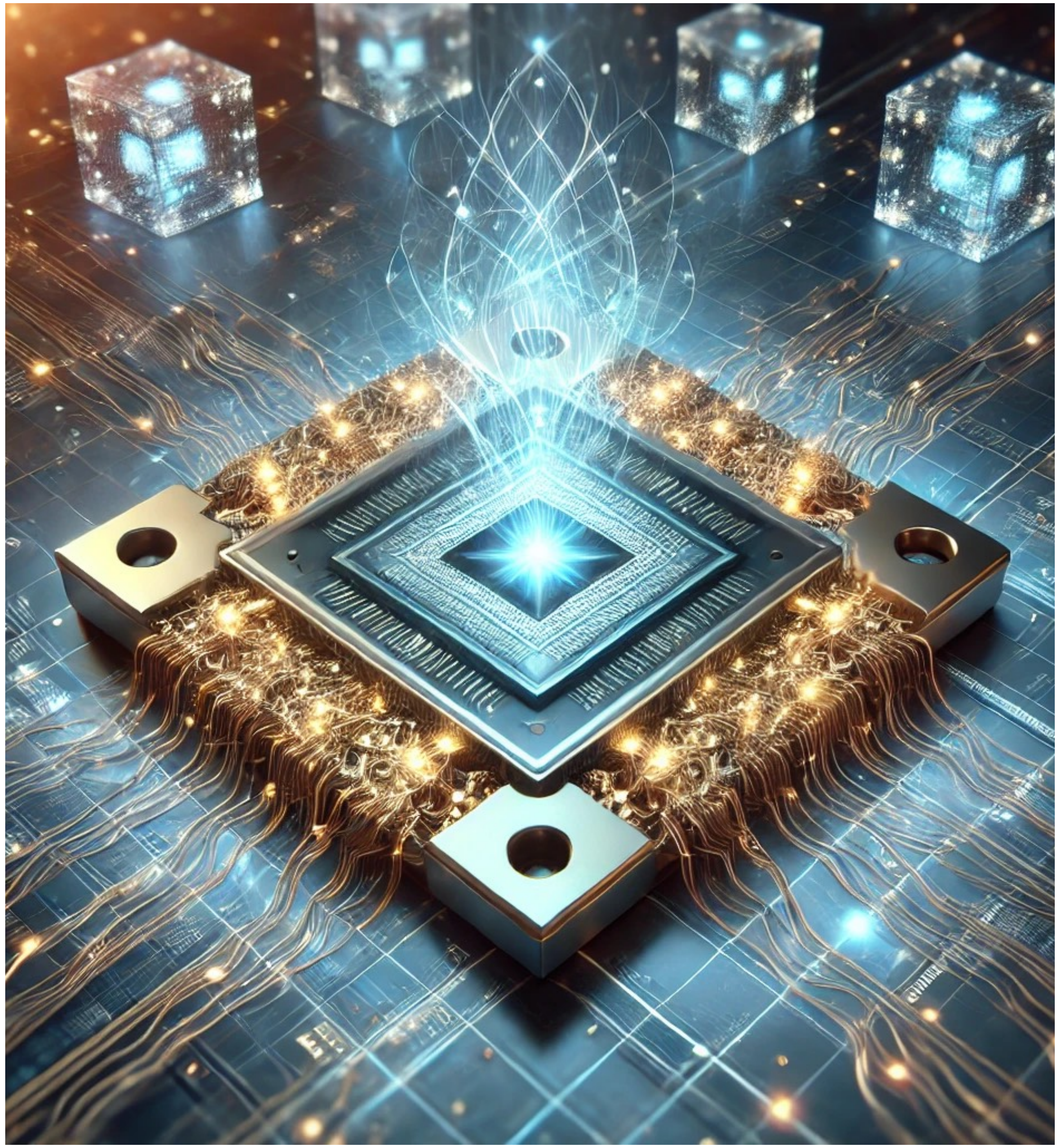
A Quantum Circuit for the Cyclic String Matching Problem

(joint work with Arianna Pavone - UniPA)



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The cyclic string matching problem

Input:

- a string $x = x_0 x_1 \cdots x_{m-1}$ of length m (pattern);
- a string $y = y_0 y_1 \cdots y_{n-1}$ of length n (text).

$x, y \in \Sigma^\star$ and $m \ll n$.

Objective: find any occurrence of any cyclic shifting of the pattern within the text,

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$x, y \in \Sigma^\star$ and $m \ll n$.

Objective: find any occurrence of any cyclic shifting of the pattern within the text,

i.e., find any $s \in \{0,\ldots,m-1\}$, $j \in \{0,\ldots,n-1\}$ such that

$$R_s(x) := x_sx_{s+1}\cdots x_{n-1}x_0\cdots x_{s-1} = y_jy_{j+1}\cdots y_{j+n-1}\,.$$

The cyclic string matching problem

Example:

- $x =$ AGGCA;
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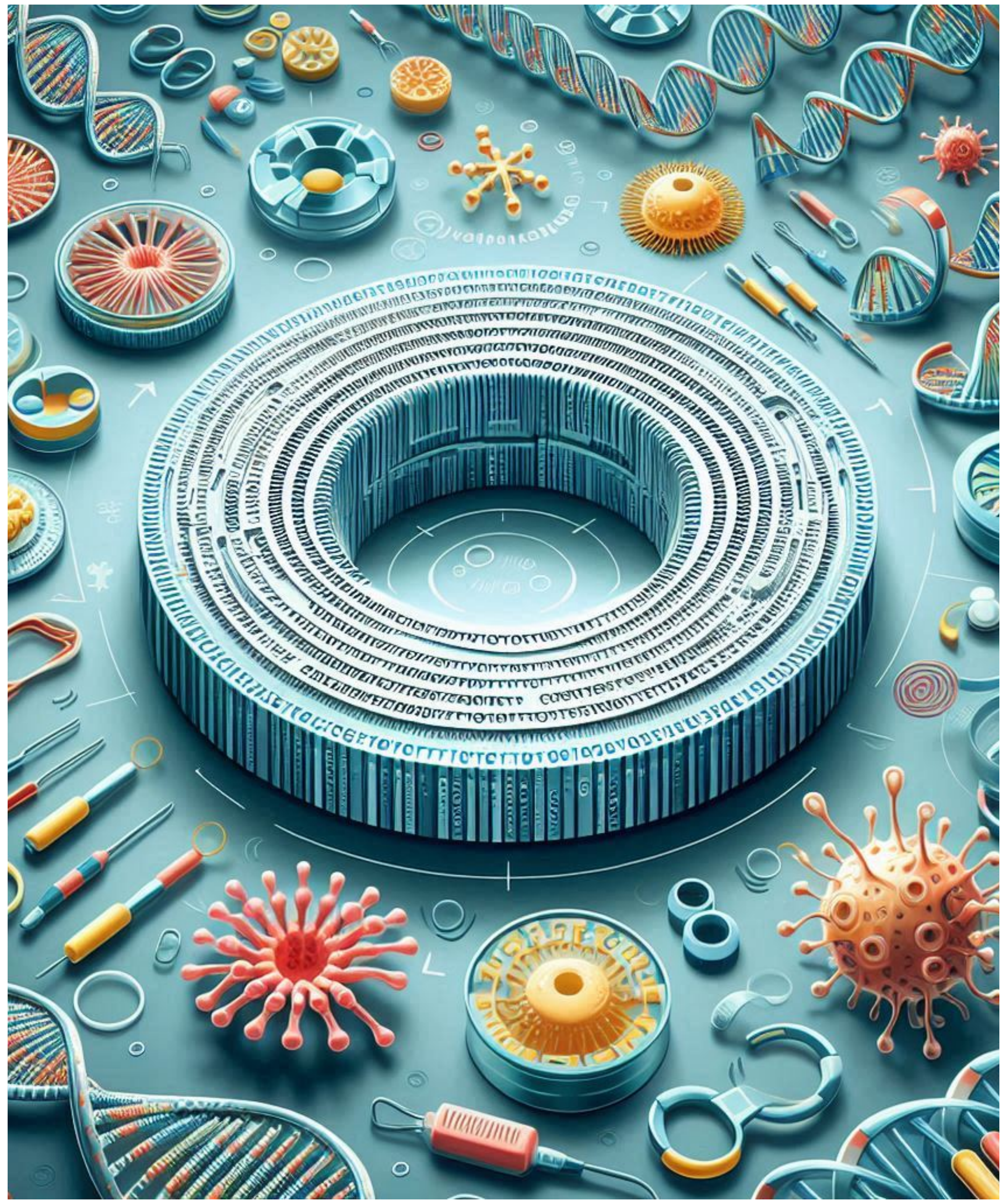
A solving algorithm should return $s = 3$ and $j = 9$.

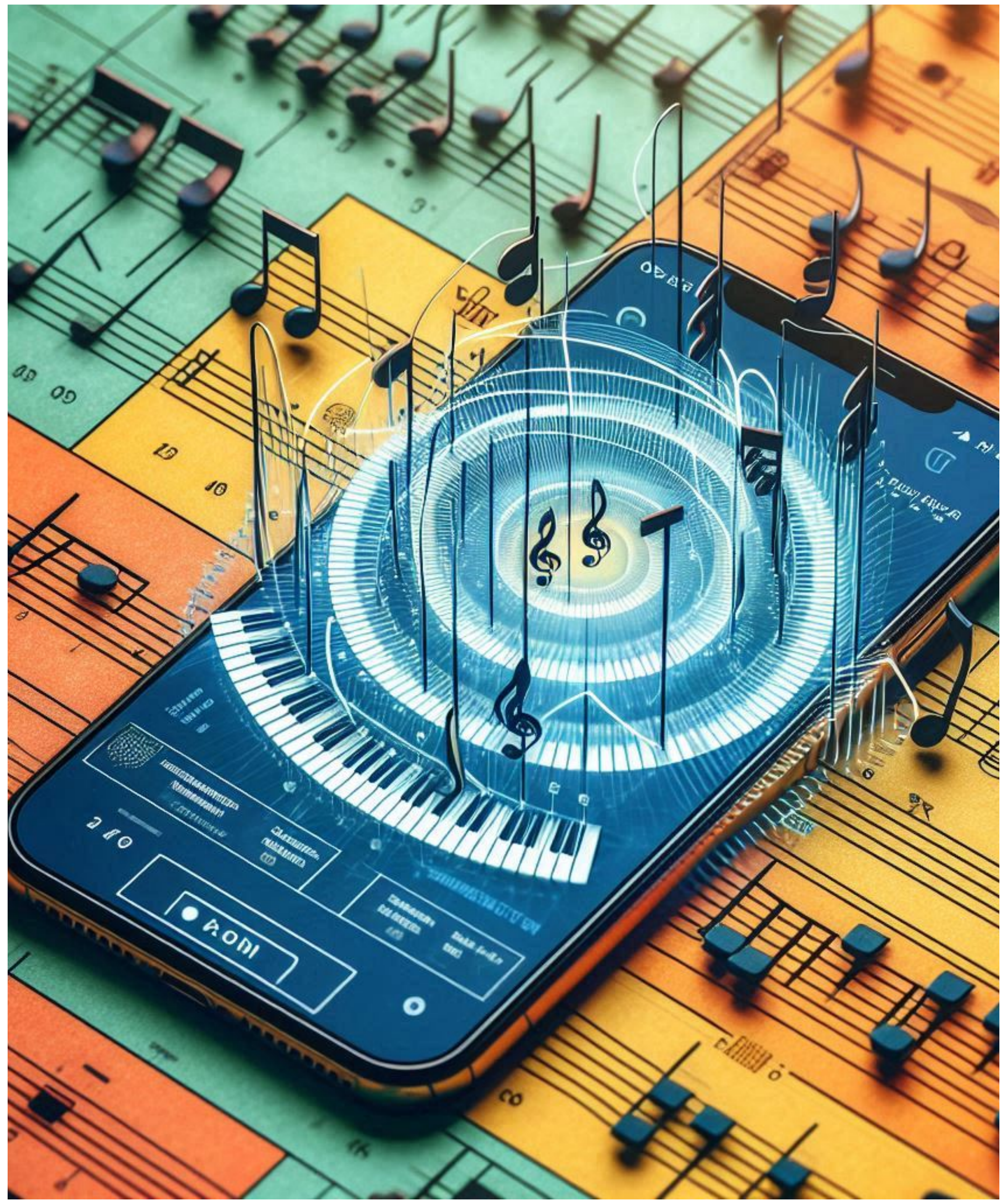
The cyclic string matching problem

A few examples of application

Pattern detection for frequency analysis of encrypted data.



Detecting circular DNA sequences like plasmids and viruses.



Identification of chord progressions or rhythm sequences for music and audio processing.

The most-efficient classical solution

The best-known classical solution is the $\mathcal{O}(n)$-Time algorithm by Lothaire (2005).

- Preprocess x by constructing a suffix automaton of the string xx;
- feed y into the automaton;
- the lengths of the longest factors of xx occurring in y can be found by the links followed in the automaton in time $\mathcal{O}(n)$.

Our result

We designed a quantum algorithm for the cyclic string matching problem.

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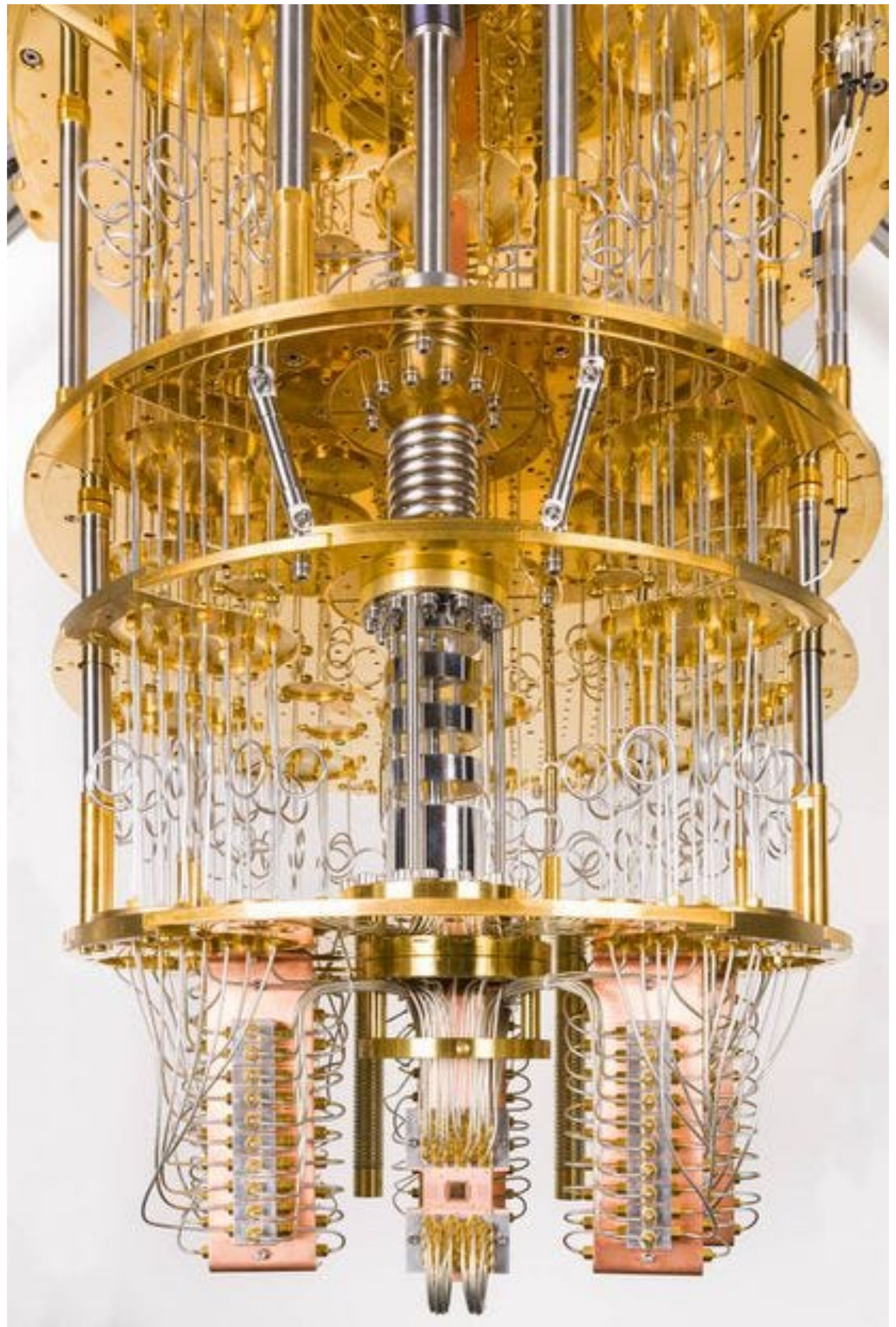
Our algorithm is formalised as a quantum circuit with

- $\tilde{\mathcal{O}}(\sqrt{n})$-Depth;
- $\mathcal{O}(n)$-Size.

It requires quadratically fewer time-steps than the most efficient counterpart algorithm running on a classical machine.

Quantum computing

Quantum algorithms are designed to take full advantage of the computational power of quantum machines.



A quantum computing machine works as predicted by the theory of quantum mechanics.

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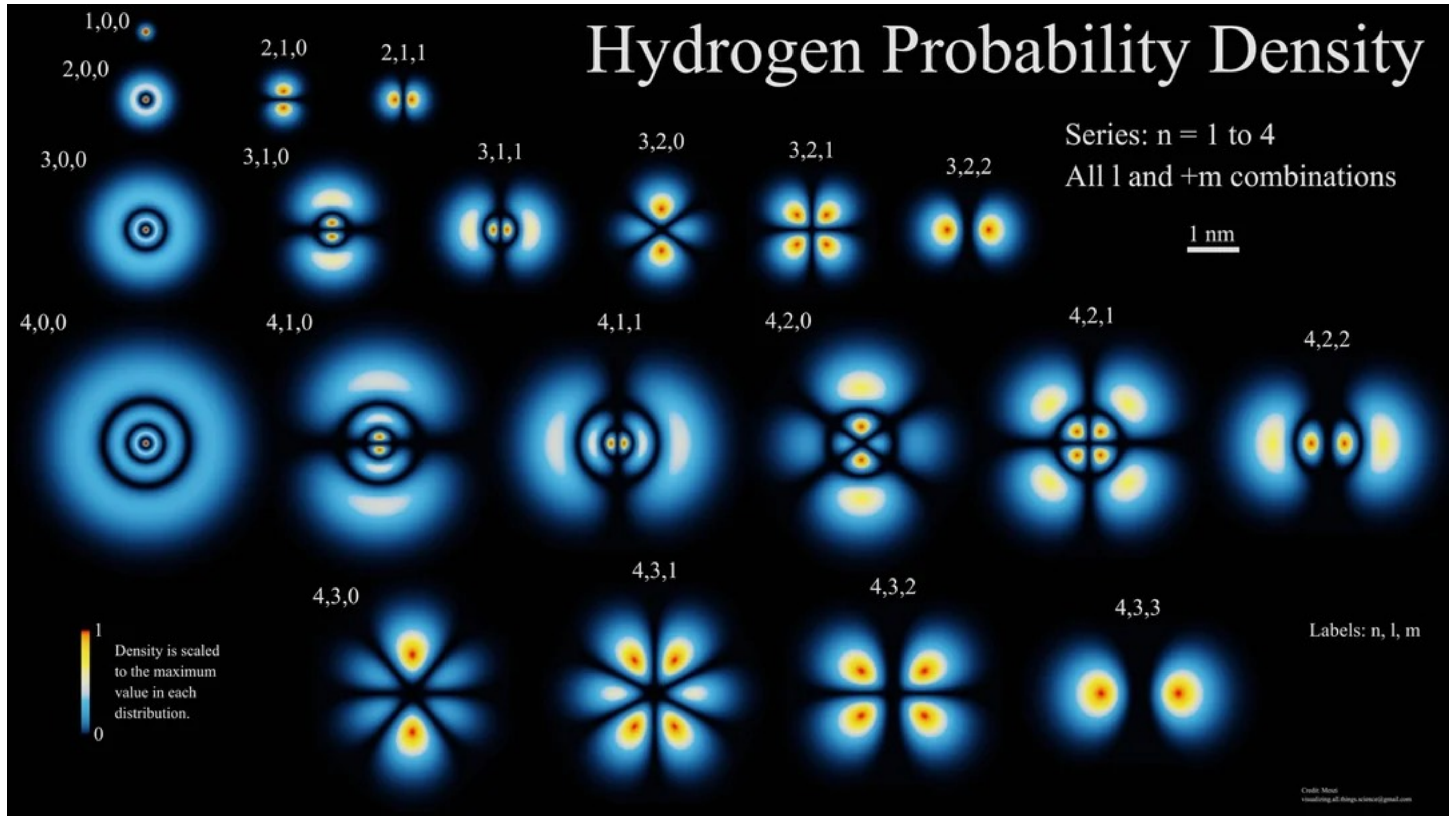
The state of each of such particles is described by a wave function, which can be thought of as a probability distribution of the classical states in which the particle can be observed.



Quantum bits

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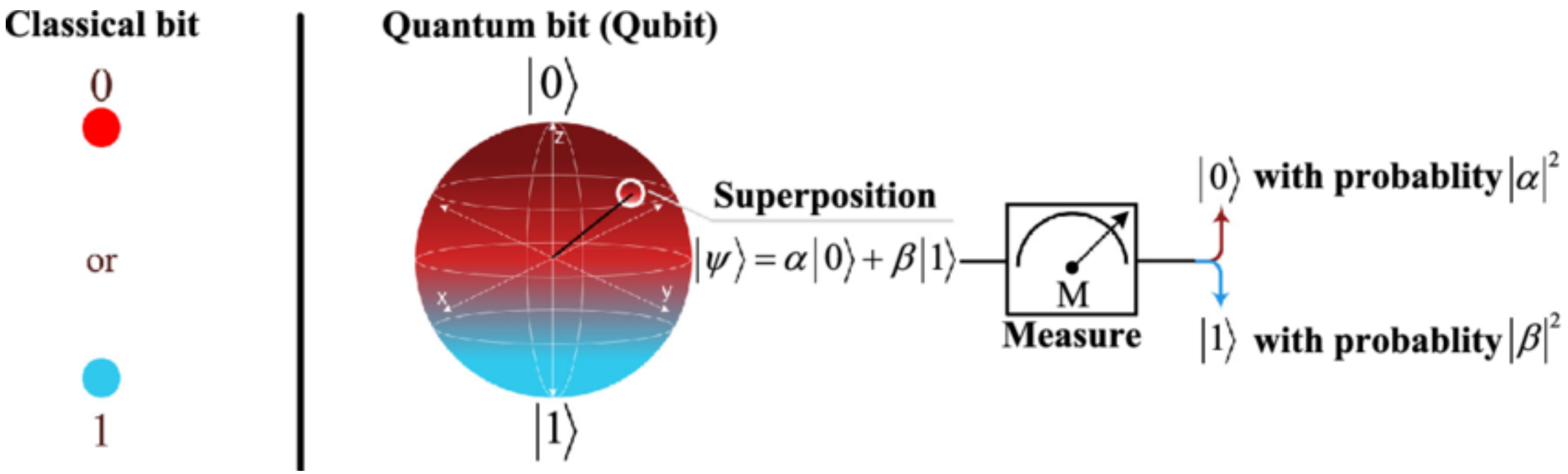
$$|q\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad \alpha, \beta \in \mathbb{C}: \ |\alpha|^2 + |\beta|^2 = 1.$$

Mathematically, a qubit $|q\rangle$ is an element from the 2-dimensional Hilbert space, $\mathscr{H}$, over the complex field.



Multiple systems of qubits

A multiple system of qubits taken together is referred to as a quantum register.

$$|k\rangle = |q_0, q_1, \ldots, q_{n-1}\rangle = \sum_{j=0}^{2^n-1} \alpha_j |k_j\rangle,$$

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- where $\alpha_j \in \mathbb{C}$: $\sum_{j=0}^{2^n-1} |\alpha_j|^2 = 1$,
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Two qubits can be entangled, that is the wave functions describing their respective states can be correlated and, therefore, not independent. Mathematically,

$$|k\rangle \in \bigotimes_{i=0}^{n-1} \mathscr{H} \, .$$

Quantum operations

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Methematically, a n-ary quantum operation on a n-qubits register is a unitary transformation, i.e., a linear bounded operator $U\colon \mathscr{H}^n \to \mathscr{H}^n$ such that $U^\dagger U = UU^\dagger = I$, where $U^\dagger$ is the complex conjugate of the transpose of U.

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There is a great variety of quantum operations, however each of them can be obtained as the composition of at most binary quantum operations.

Exploiting quantum mechanics to solve the cyclic string matching problem

Example:

Pattern: bac

Text: $abbac$

$m = 3, n = 5$

a	b	b	a	c
b	b	a	c	a
b	a	c	a	b
a	c	a	b	b
c	a	b	b	a

b	a	c
a	c	b
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is it a match ?

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The idea is to harness quantum mechanics and check wether the first m qubits of the register containing the superposition of all possible cyclic shiftings of the text matches the register containing the superposition of all possible cyclic shiftings of the pattern.

Our contribution

Our contribution is a refinement of an algorithm from Niroula&Nam's for exact string matching.

Our algorithm uses a well known quantum algorithm that searches for a desired item within an unstructured database of items: Grover's search algorithm (1996).

There are several models of quantum computation. The model we adopt is that of quantum circuits.

Quantum circuits

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- Each gate operate an elementary operation on at most two qubits.
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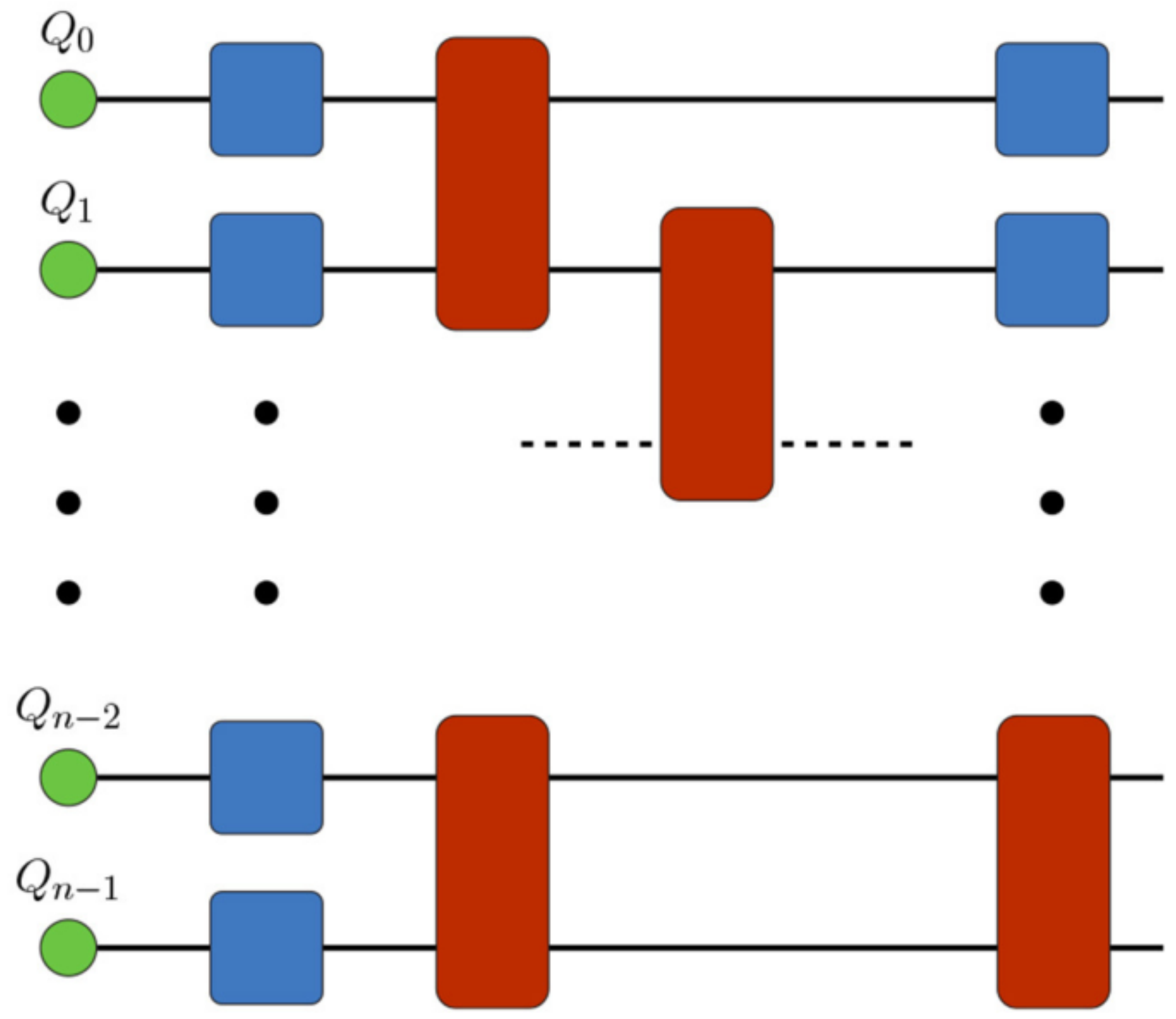
A quantum circuit can be seen as a sequence of parallel wires (each corresponding to a qubit) passing through certain gates that operate on them.



Quantum circuits

We measure the complexity of a quantum circuit by its **depth** (corresponding to the number of time-steps needed).

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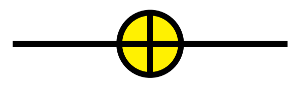

$$|0\rangle \mapsto |1\rangle, \quad |1\rangle \mapsto |0\rangle$$

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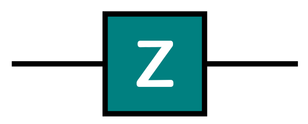
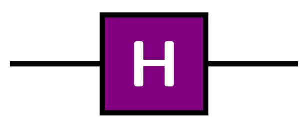


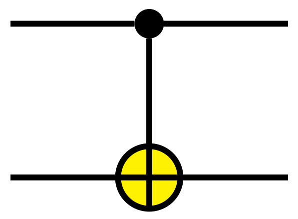
$$|0\rangle \mapsto |0\rangle, \quad |1\rangle \mapsto -|1\rangle$$

Hadamard



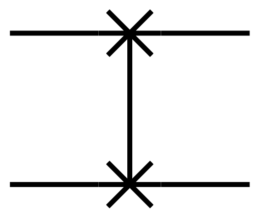
$$|0\rangle \mapsto \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle, \quad |1\rangle \mapsto \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = |-\rangle$$

CNOT



$$|q_0, q_1\rangle \mapsto |q_0, q_0 \oplus q_1\rangle$$

Swap



$$|q_0, q_1\rangle \mapsto |q_1, q_0\rangle$$

Grover's search algorithm

It searches a black-box list of N items in $\tilde{\mathcal{O}}(\sqrt{N})$-time, whereas classical algorithms need $\mathcal{O}(N)$-time.

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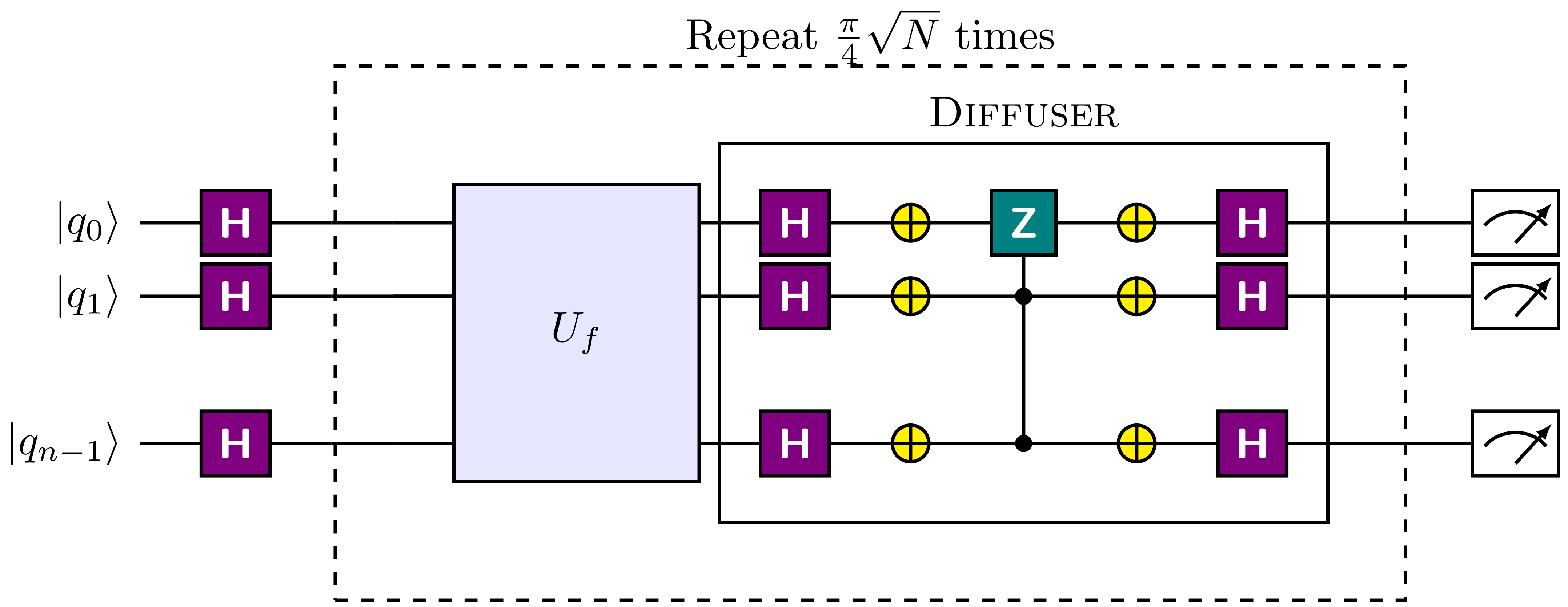
Let $f\colon \{0,1\}^n \to \{0,1\}$ such that $\exists!\ x^\star : f(x^\star) = 1$.
We have to search in a space of $N = 2^n$ items.



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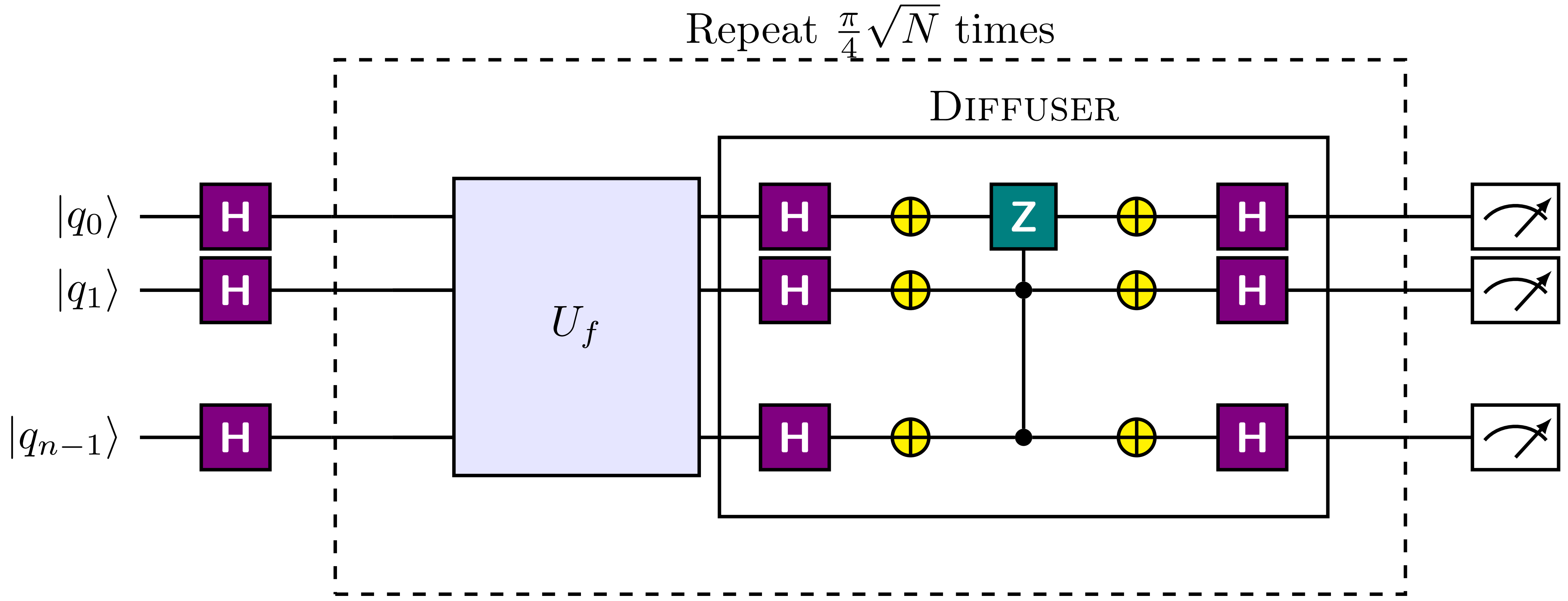

- Initialisation: we start with the register $|0\rangle^{\otimes n}$, in which each qubit initialized to $|0\rangle$, and apply the Hadamard gate to each qubit, obtaining a superposition of all possible items $x_0, \ldots, x_{N-1}$, i.e. $\sum_{i=0}^{2^n-1} \frac{1}{\sqrt{2^n}} |x_i\rangle$.

- Iterative phase (performs a rotation of $\frac{2}{\sqrt{N}}$ rads):
  - Phase oracle: $U_f|x\rangle = (-1)^{f(x)}|x\rangle$. It flips the phase of $|x^\star\rangle$.
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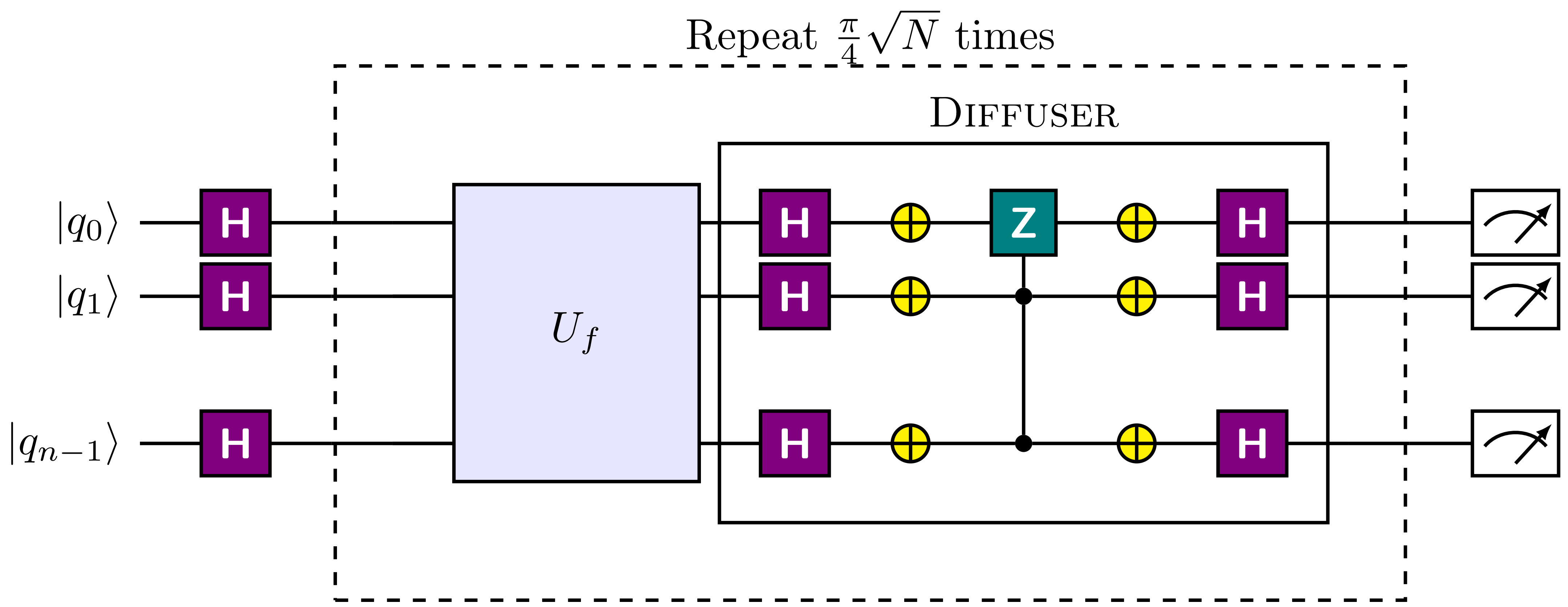

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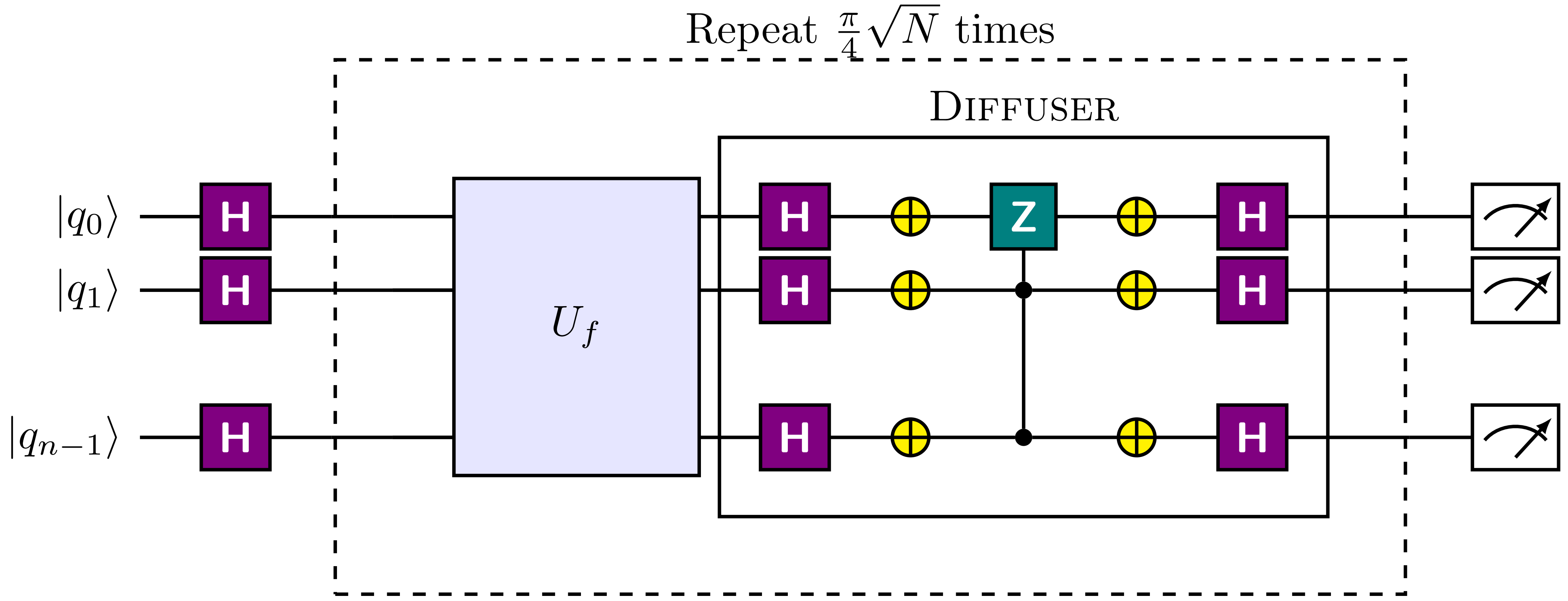

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Depth: $\mathcal{O}\left(\sqrt{N}\,\left(T(n) + \log(n)\right)\right)$, where $T(n)$ is the depth of the phase oracle, and $\log(n)$ is the depth of the multi-controlled Z gate.

A quantum circuit for cyclic string matching

Input: $|\varphi\rangle = |i\rangle \otimes |j\rangle \otimes |x\rangle \otimes |y\rangle$.

- $|x\rangle$: m-qubit register containing the characters from the pattern.
- $|y\rangle$: n-qubit register containing the characters from the text.
- $|i\rangle = |0\rangle^{\otimes \log(m)}$.
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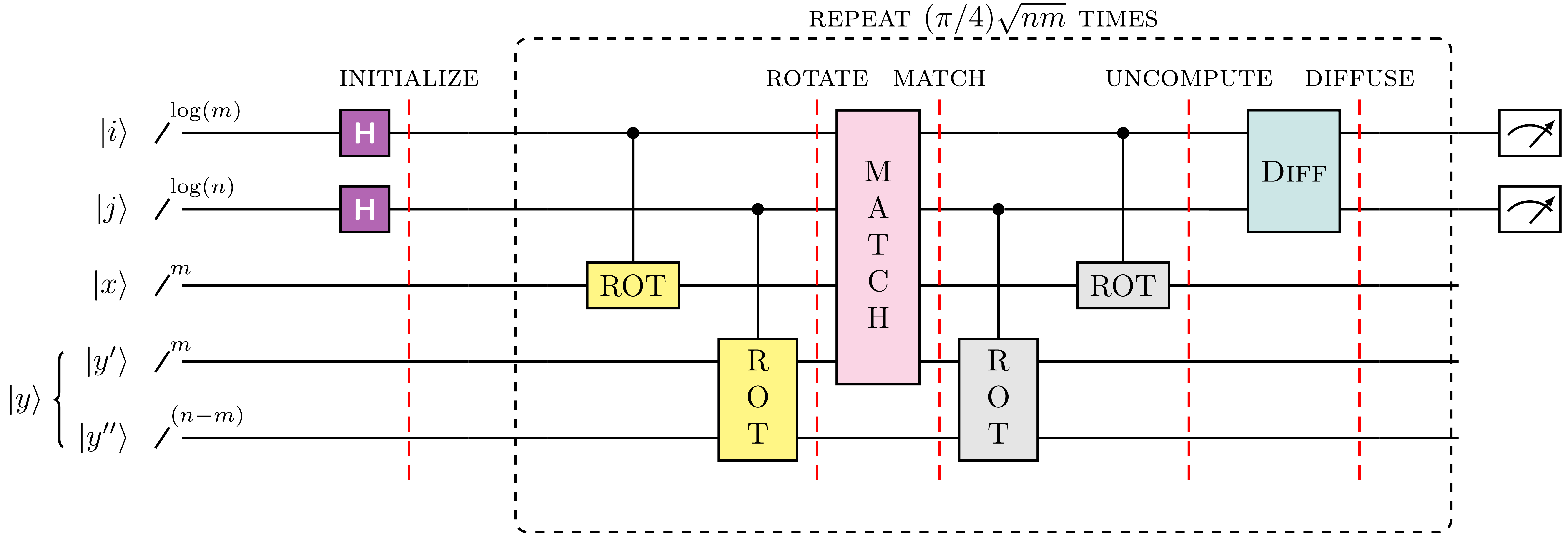

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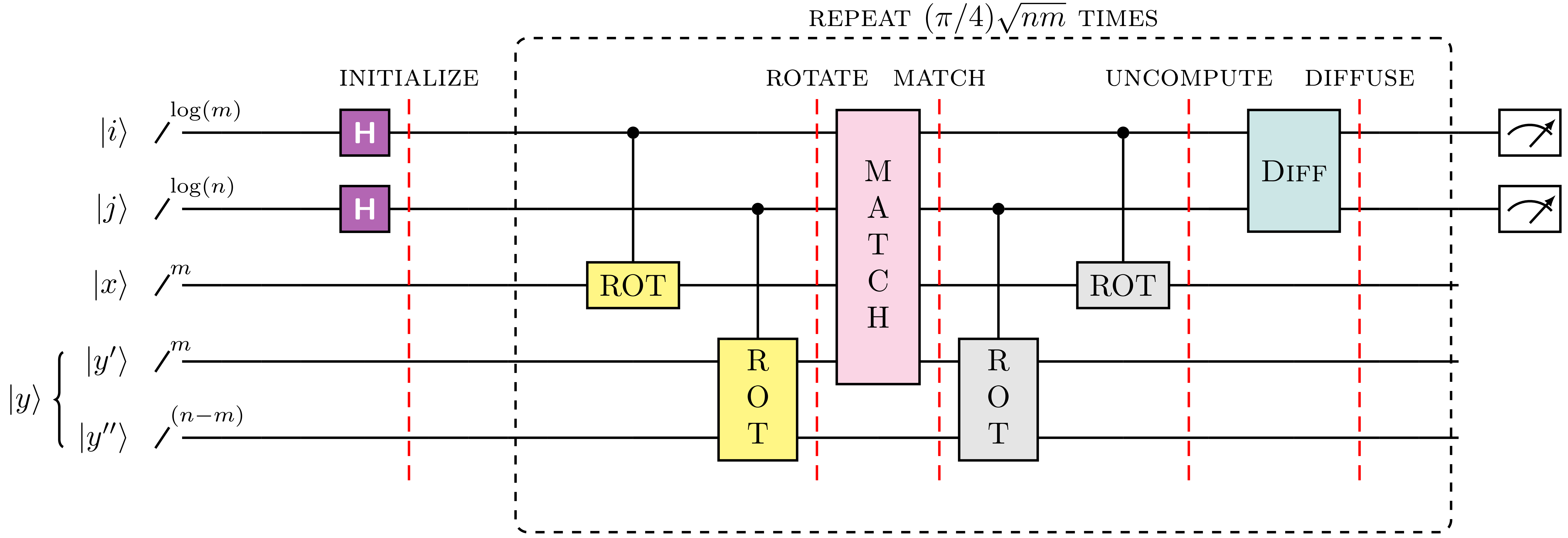

During the initialisation, the algorithm applies the Hadamard operator (H) to all the qubits in $|i\rangle$ and $|j\rangle$ so that their states are the superpositions of all possible shift values between 0 and $m-1$ and between 0 and $n-1$, respectively.

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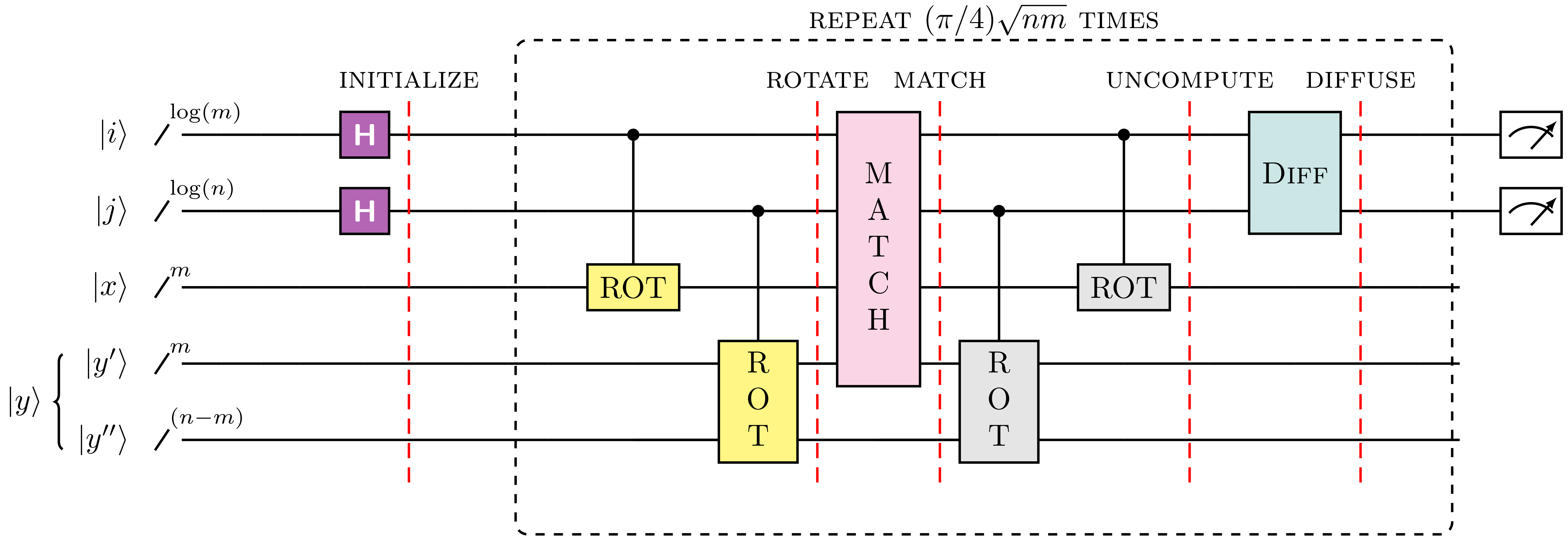

ROT: the controlled cyclic shift operator. After its application, the registers $|x\rangle$ and $|y\rangle$ are in the state of superposition of all their respective cyclic shiftings.

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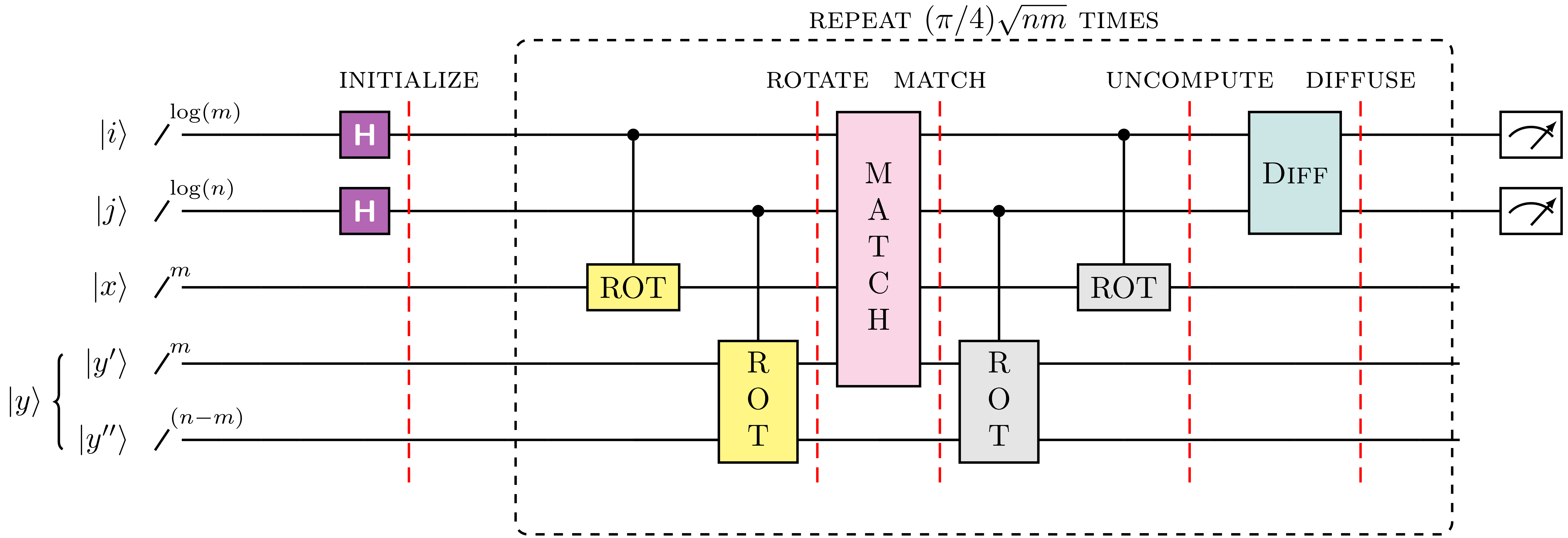

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MATCH: a phase oracle for exact string matching, i.e., for the function $f\colon \{0,1\}^m \times \{0,1\}^m \to \{0,1\}$ such that

$$f(x, y[0..m-1]) = \begin{cases} 1 & \text{if } x_i = y_i \text{ for all } 0 \le i < m \\ 0 & \text{otherwise.} \end{cases}$$

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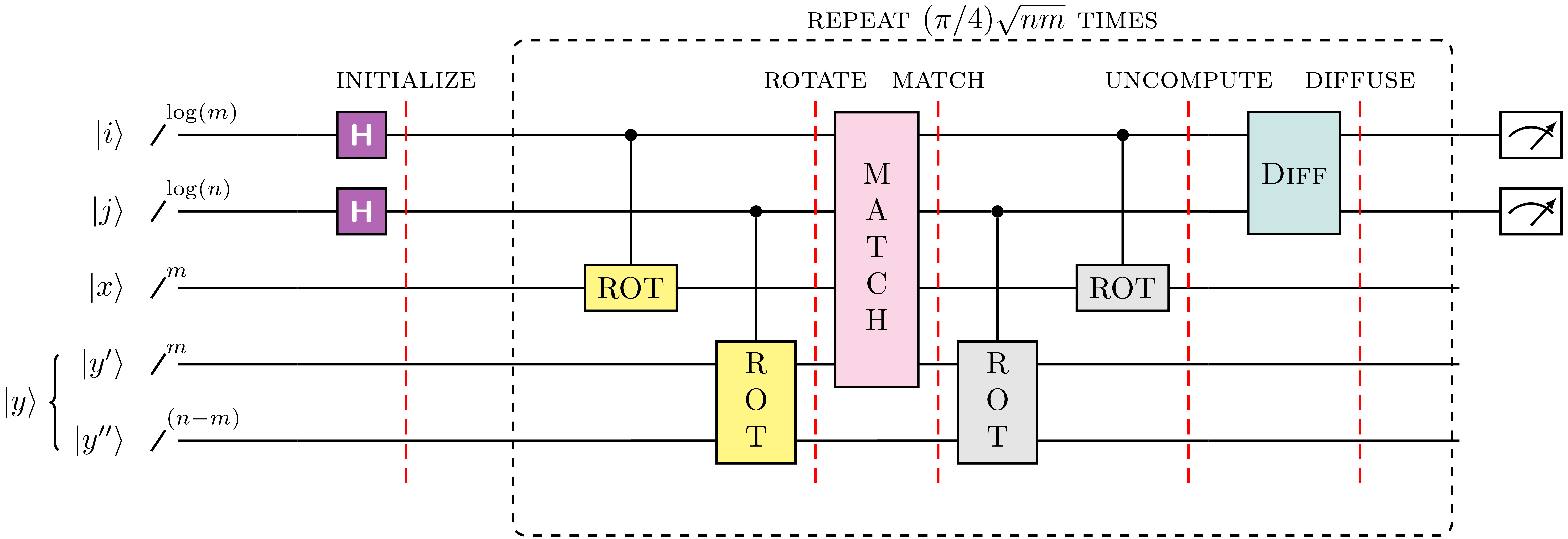

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DIFF: Grover's diffuser.

The cyclic shifting operator

Given an input n-qubit register $|q\rangle$, a **cyclic shift operator** R_s applies a rightward shift of s positions.

Formally,

$$|q_0, q_1, \ldots, q_{n-1}\rangle \mapsto |q_s, q_{s+1}, \ldots, q_{n-1}, q_0, q_1, \ldots, q_{s-1}\rangle\,.$$

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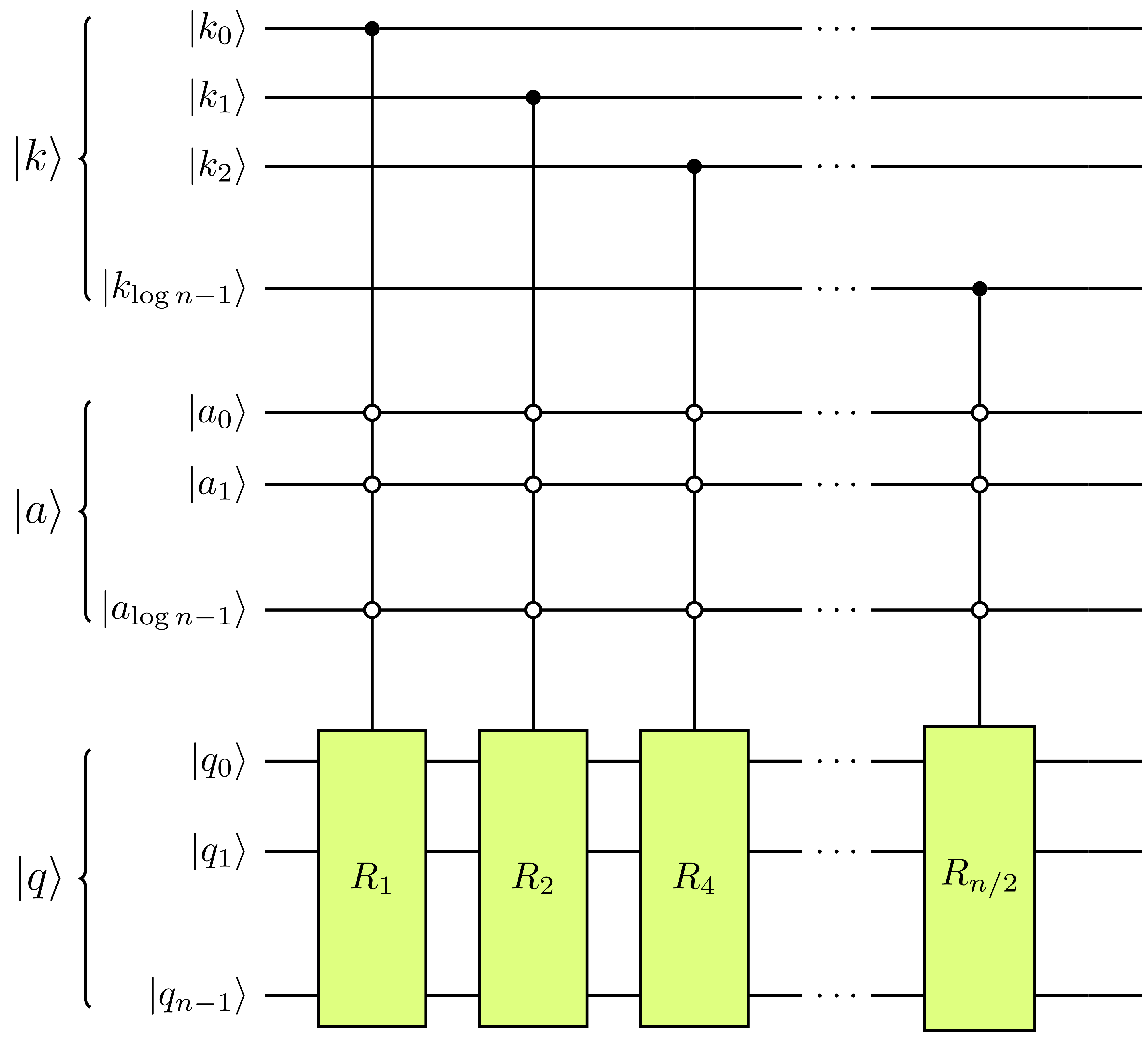
For every (n, s) , there exists a $\mathcal{O}(\log(n))$**-depth** quantum circuit that applies R_s to an input n-qubit register.

Such a circuit run in at most $\log(n)$ time-steps; the jth time-step consists of $\frac{n}{2^{j+1}}$ parallel swaps.

The controlled cyclic shifting operator

ROT creates a superposition of all circular rotations of the text y. It **applies to an input n-qubit register $|q\rangle$, a cyclic shifting of a number of positions that depends on a second input register $|k\rangle$ of length $\log(n)$.**



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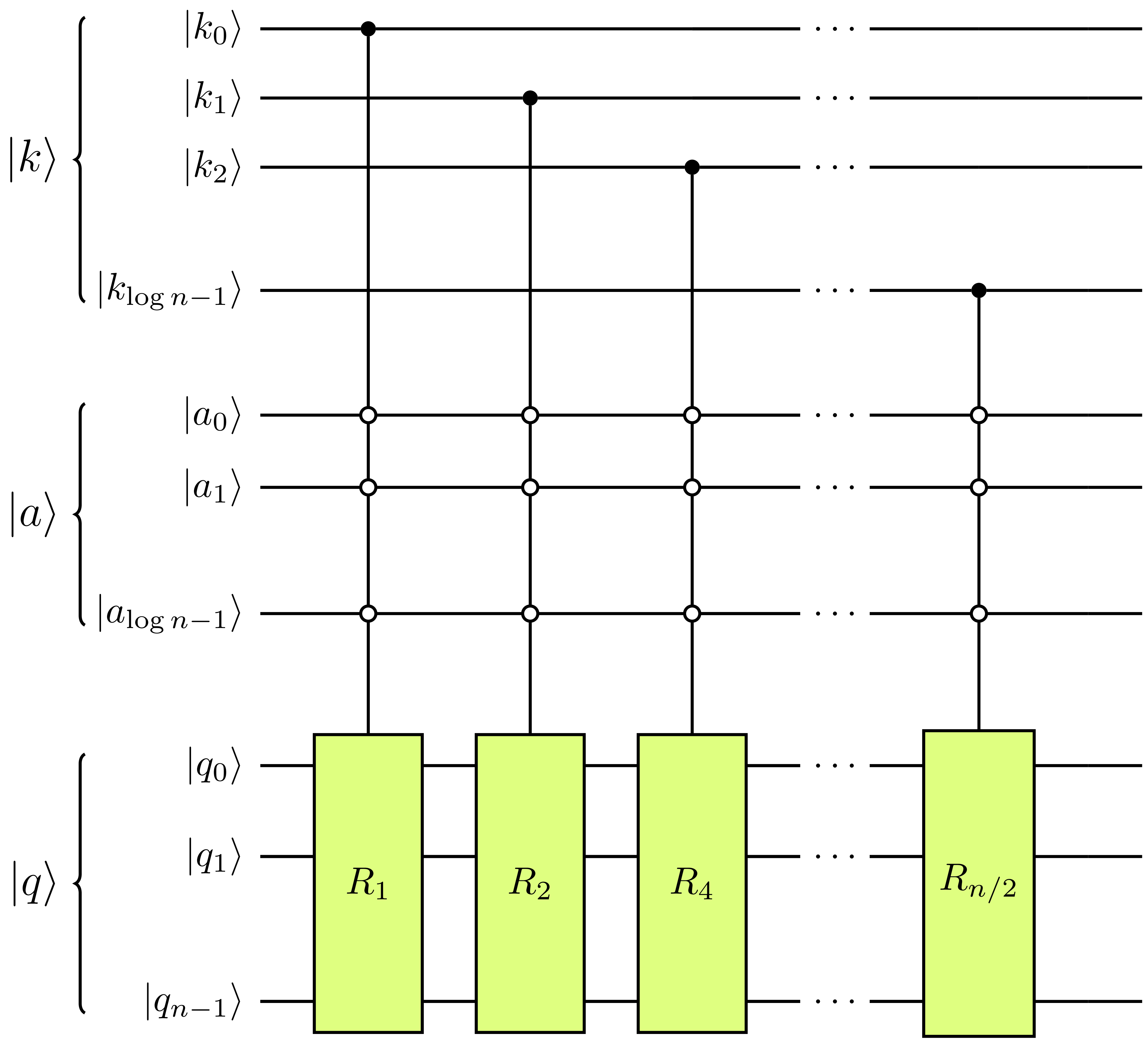
Since $|k\rangle = \bigotimes_{i=0}^{\log(n)-1} |k_i\rangle$, the operator ROT can be implemented by applying to $|q\rangle$ the operator R_{2^i} controlled by $|k_i\rangle$, for each $0 \leq i < \log(n)$. It requires $\log(n)$ ancilla qubits.



The controlled cyclic shifting operator

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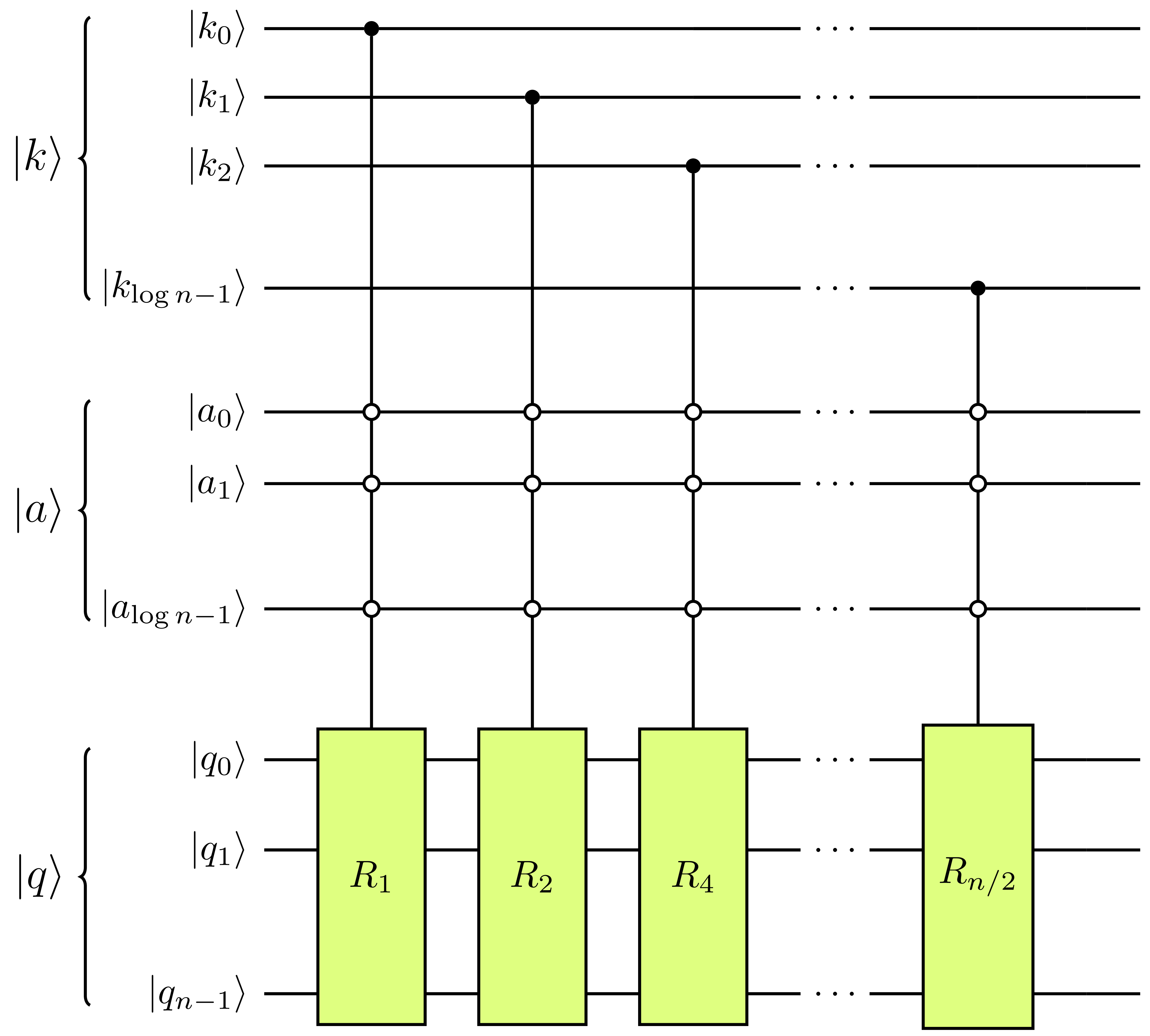
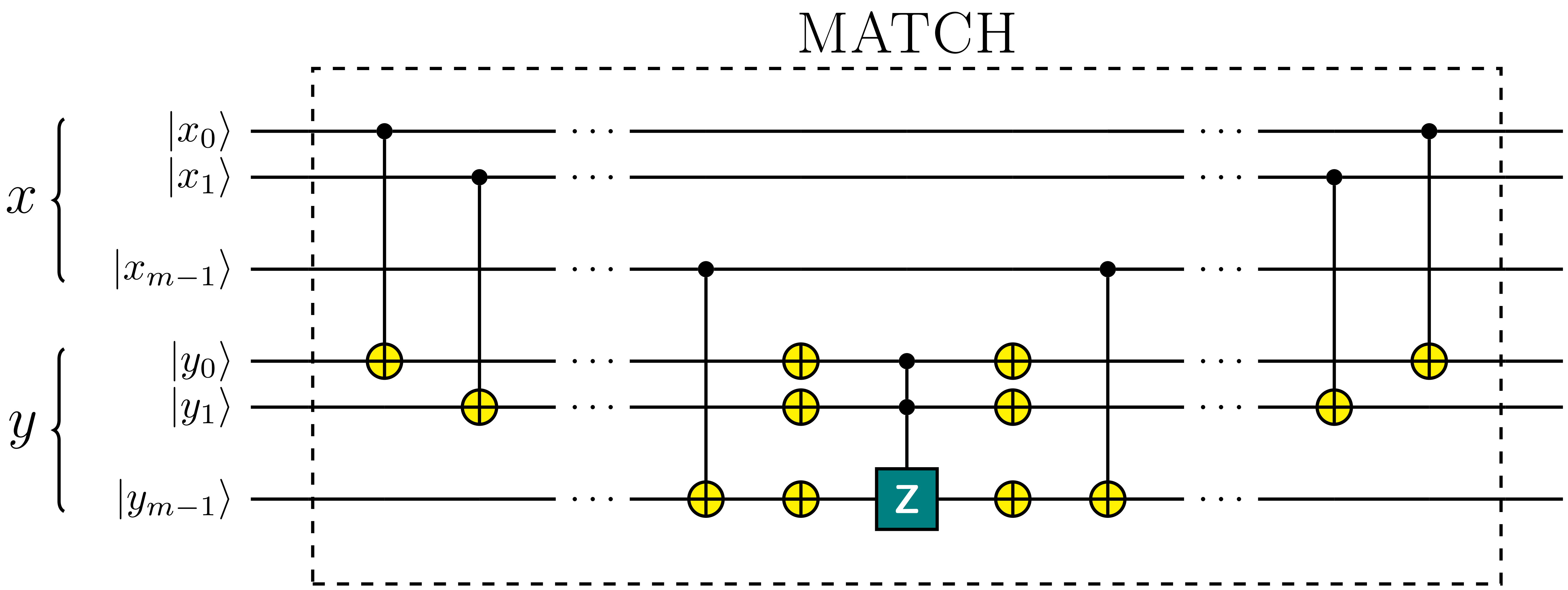
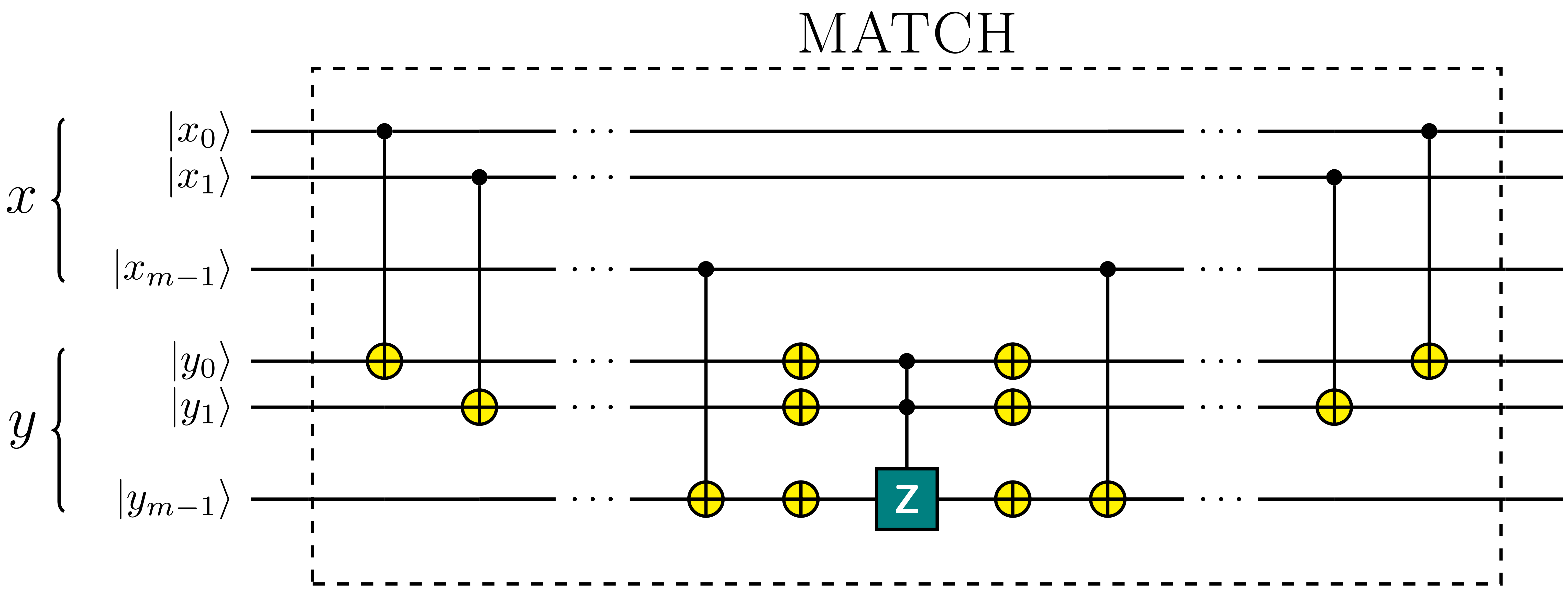
The **depth** of such a circuit implementing ROT is $\mathcal{O}(\log^2(n))$.



A phase oracle for exact string matching



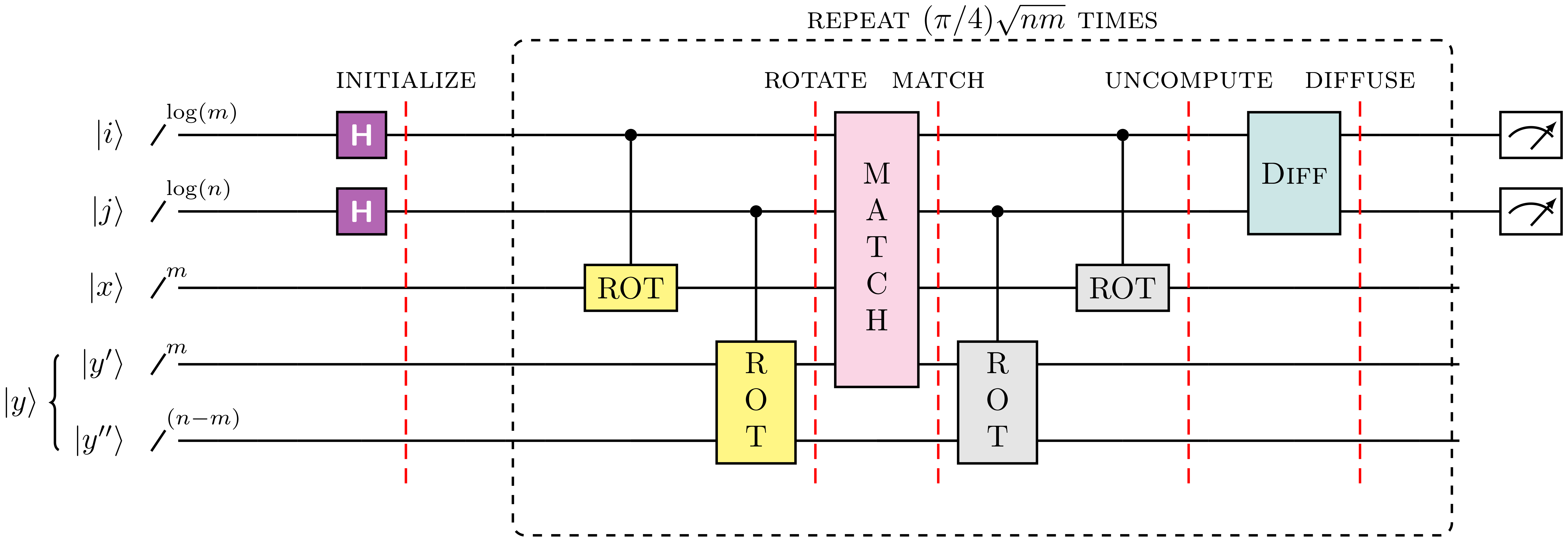
A phase oracle for exact string matching



- The n CNOTs can run in parallel as well as the n X gates.
- The multi-controlled flip-phase (Z) gate takes $\mathcal{O}(\log(m))$ time.

Depth: $\mathcal{O}\left(\log(m)\right)$.

A quantum circuit for cyclic string matching



ROT: the controlled cyclic shift operator. After its application, the registers $|x\rangle$ and $|y\rangle$ are in the state of superposition of all their respective cyclic shiftings.

Depth: $\mathcal{O}\left(\log^2(m)\right)$ and $\mathcal{O}\left(\log^2(n)\right)$, respectively.

MATCH: a phase oracle for exact string matching, i.e., for the function $f\colon \{0,1\}^m \times \{0,1\}^m \to \{0,1\}$ such that

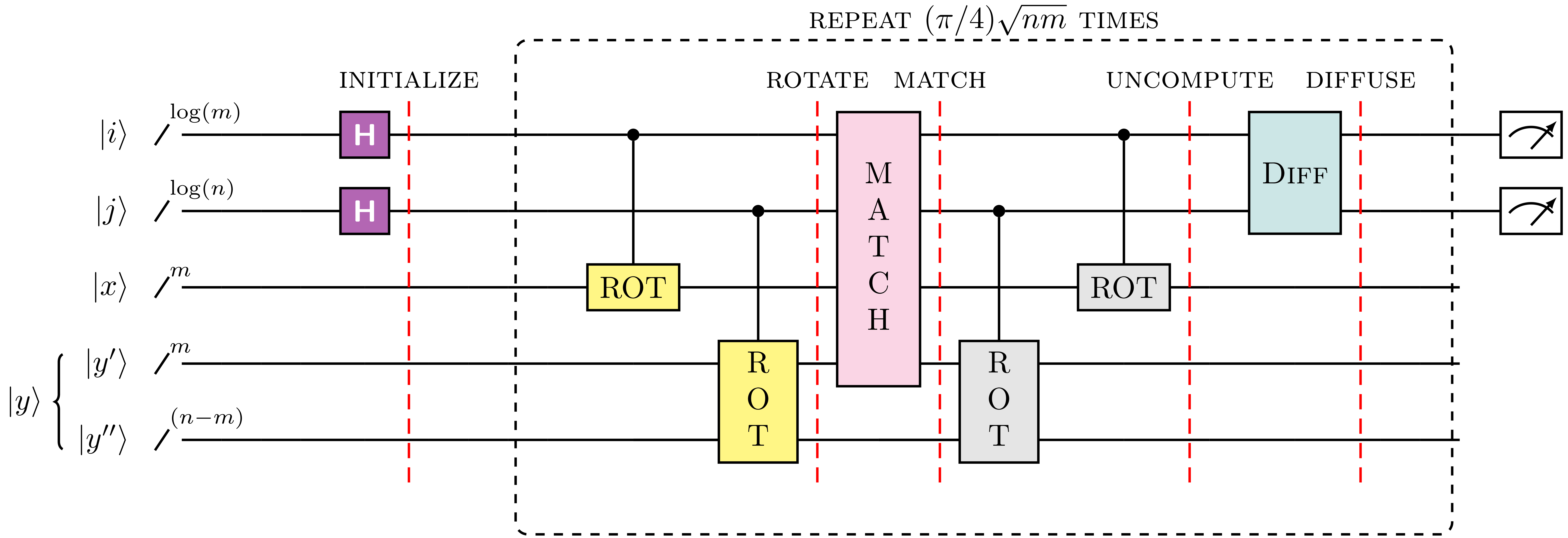
$$f(x, y[0..m-1]) = \begin{cases} 1 & \text{if } x_i = y_i \text{ for all } 0 \le i < m \\ 0 & \text{otherwise.} \end{cases}$$

Depth: $\mathcal{O}\left(\log(m)\right)$.

DIFF: Grover's diffuser.

Depth: $\mathcal{O}\left(\log(\log(n))\right)$.

A quantum circuit for cyclic string matching



<u>**Overall depth**</u>: $\mathcal{O}\left(\sqrt{n}\left(\log^2(m)+\log^2(n)+\log(m)+\log(\log(n))\right)\right)=\mathcal{O}\left(\sqrt{n}\left(\log^2(n)\right)\right)=\tilde{\mathcal{O}}\left(\sqrt{n}\right).$

<u>**Space**</u> : $\mathcal{O}\left(n+m\right)=\mathcal{O}\left(n\right)$. <u>**Size**</u>: $\mathcal{O}\left(n\log(n)\right)$.

Research perspective

- Can we improve this circuit to get a better performance (i.e., smaller depth or less qubits) so that it would be implementable on near-terms utility-scale quantum hardware?

- Quantum algorithms for approximate versions of cyclic shifting matching problem which find application in bioinformatics, etc.

- Quantum algorithms for Lyndon words and Lyndon factorisation.

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Thanks

Ps. You can play with our codes written in qiskit at
https://colab.research.google.com/drive/1bbRjsYl7UCVT6P4gNwJulARfd64L1RCT?usp=sharing