Computational Substantiation of the *d*-step Conjecture for Distinct Squares Revisited

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Outline

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Motivation and background

- In a pivotal paper in 1981, *Crochemore* showed that the maximum number of maximal repetitions in a string is $O(n \log(n))$, attained by Fibonacci strings.
- *Maximal repetitions*, a precursor to *runs*, may contain several squares bundled up, so bounding the maximum number of squares is a different problem.
- In 1998, in another pivotal paper, *Fraenkel* and *Simpson* showed that the number of occurrences of squares is bounded by $n \log_\Phi(n) \approx 1.441 n \log_2 2(n)$ (Φ denotes the golden ratio).
- Improved in 2020 by Bannai et. al to $n \log_2(n)$.

- The main result by *Fraenkel* and *Simpson* in their 1998 paper is that the maximum number of **distinct squares**, when types rather than occurrences are counted, is bounded by $2n$. They conjectured that the bound should be $\leq n$.
- The most significant aspect of their work was the fact that only 2 rightmost squares can start at the same position.
- The combinatorics analysis of so-called double squares was pioneered by *Lam*, and fully developed by *Deza*, *Franek*, and *Thierry* in 2015, giving an upper bound for MNDS[1] as $\frac{11}{6}n \approx 1.83n$.

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[1] MNDS=maximum number of distinct squares

- Since then, several partial results concerning the densities of distinct squares distribution had been published – *Blanchet-Sadri et. al* and *Manea et. al*.
- In 2011, *Deza*, *Franek*, and *Jiang* presented the d-step approach to the problem and conjectured the bound to be $\leq n-d$ where d is the number of distinct symbols in the string (d-step conjecture).
- In 2012 they introduced *a computational substantiation of the d-step approach to the number of distinct squares problem* that allowed to approximately double the length of the strings for computational verification of MNDS conjecture and d-step conjecture.

- Note that the problem has two versions – counting all distinct squares, or a simpler version of counting all distinct *primitively rooted* squares.
- Fraenkel+Simpson's and Deza+Franek+Thierry's results are for all distinct squares, while Deza+Franek+Jiang were formulated for all distinct primitively rooted squares.
- There does not seem to be any essential reason not to be able to reformulate Deza+Franek+Jiang's result for all distinct squares, however the posted results are for primitively rooted version as the software used counted only the primitively rooted squares.

$(d, n{-}d)$ table

A string $\boldsymbol{x}$ is a (d, n)-string if it has d distinct symbols and its length equals n

abba is a $(2, 4)$-string (the distinct symbols are *a* and *b*)

abcabdbabd is a $(4, 10)$-string (the distinct symbols are *a*, *b*, *c*, and *d*)

$s(\boldsymbol{x}) =$ *the number of distinct squares in string* $\boldsymbol{x}$

$$\sigma_d(n) = \max\{s(\boldsymbol{x}) \ : \ \boldsymbol{x} \text{ is a } (d, n)\text{-string}\}$$

The $(d, n-d)$ table is an infinite table that contains the values $\sigma_d(n)$ for all $n \geq 2$ and all $2 \leq d \leq n$.

Normally, it would be expected to be organized in rows indexed by d and columns indexed by n:

		n					
		2	3	4	....	k	....
d	2			$\sigma_2(4)$			
	3						
	4						
	$\vdots$						
	r					$\sigma_r(k)$	
	$\vdots$						

In the $(d, n-d)$ table, we organize the entries differently, the rows are again indexed by d, but the columns are indexed by $n-d$:

	n-d						
d		2	3	4	....	k	....
	2			$\sigma_2(6)$			
	3						
	4						
	⋮						
	r					$\sigma_r(k+r)$	
	⋮						

n - d

d	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
2	2	2	3	3	4	5	6	7	7	8	9	10	11	12	12	13	13	14	15	16	17	18	19	20	20	21	22	23	23	23	24	25	26	27
3	2	3	3	4	4	5	6	7	8	8	9	10	11	12	13	13	14	14	15	16	17	18	19	20	21	21	22	23	24	24	25	26	26	27
4	2	3	4	4	5	5	6	7	8	9	9	10	11	12	13	14	14	15	15	16	17	18	19	20	21	22	22	23	24	25	25	26	27	
5	2	3	4	5	5	6	6	7	8	9	10	10	11	12	13	14	15	15	16	16	17	18	19	20	21	22	23	23	24	25	26			
6	2	3	4	5	6	6	7	7	8	9	10	11	11	12	13	14	15	16	16	17	17	18	19	20	21	22	23	24						
7	2	3	4	5	6	7	7	8	8	9	10	11	12	12	13	14	15	16	17	17	18	18	19	20	21	22	23	24	25					
8	2	3	4	5	6	7	8	8	9	9	10	11	12	13	13	14	15	16	17	18														
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11	2	3	4	5	6	7	8	9	10	11	11	12	12	13	14	15	16	16	17	18	19	20	21											
12	2	3	4	5	6	7	8	9	10	11	12	12	13	13	14	15	16	17																
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20	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	20														
21	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21														
22	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22													
23	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23												
24	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24											

d	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68
2	28	29	30	31	32	33	33	34	35	36	36	37	37	38	39	40	41	42	43	44	44	45	46	47	48	48	49	50	51	52	53	54	55
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Deza+Franek+Jiang showed that there are a lot of relationships in the table such as:

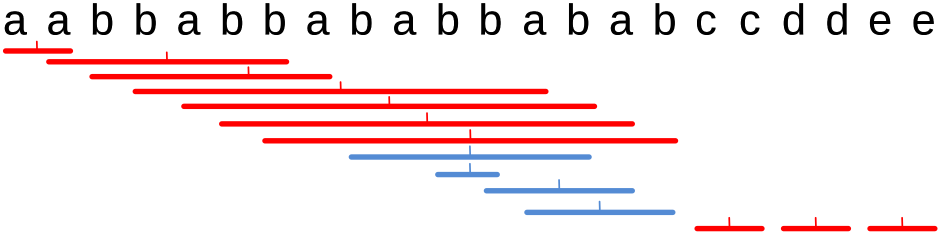
- Uniform below diagonal
  $(\forall\, 2 \le d)\big(\sigma_{d+k}(2d+k) = \sigma_d(2d)\big)$
- Diagonal rules $(\forall\, 2 \le d \le n)\big(\sigma_d(n) \le n-d\big) \Leftrightarrow (\forall\, 2 \le d)\big(\sigma_d(2d) = d\big)$
- Row increase
  $(\forall\, 2 \le d \le n)\big(\sigma_d(n) \le \sigma_d(n+1) \le \sigma_d(n)+2\big)$
- Column increase $(\forall\, 2 \le d \le n)\big(\sigma_{d+1}(n+1) \ge \sigma_d(n)\big)$
- Diagonal increase
  $(\forall\, 2 \le d)\big(\sigma_d(2d) \le \sigma_{d+1}(2d+2) \le \sigma_d(2d)+2\big)$

- The $(d, n-d)$ table entries can be filled in a fashion similar to dynamic programming, from left to right and top to bottom.
- This approach was utilized to compute the values.
- Knowing $\sigma_{d-1}(n-2)$, $\sigma_d(n-1)$, $\sigma_{d+1}(n)$, and $\sigma_{d-1}(n-1)$ strongly limits the pool of candidates for $\sigma_d(n)$.

	$\sigma_{d-1}(n-2)$	$\sigma_{d-1}(n-1)$
	$\sigma_d(n-1)$	$\sigma_d(n)$
	$\sigma_{d+1}(n)$	

S-cover

Some strings are “made of squares”:



aabbabbababbababccddee is one of the square-maximal (5,22)-strings; $\sigma_5(22) = 14$.

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Definition

A sequence $\{(a_i, b_i) : 1 \le i \le k\}$ is an **S-cover** of string $\boldsymbol{x} = \boldsymbol{x}[1..n]$, if

1. $(\forall i \in 1..k)$ $\boldsymbol{x}[a_i..b_i]$ is a rightmost square
2. $(\forall i \in 1..k-1)$ $a_i < a_{i+1}$ and $b_i < b_{i+1}$
3. $(\forall j \in 1..n)(\exists i \in 1..k)$ $a_i \le j \le b_i$
4. for every rightmost square (a, b) in $\boldsymbol{x}$, there is $i \in 1..k$ so that $a_i \le a < b \le b_i$

Observation

- *From condition 3, $a_1 = 1$ and $b_k = n$.*
- *If a string has an S-cover, then the S-cover is unique.*

Examples:

- the red squares form the S-cover of ***aabbababbababbababccddee***.
- the red squares + the first blue square is not an S-cover, the blue square violates the 2nd condition.
- ***abcabc*** has an S-cover, the S-cover consists of a single square *abcabc*.
- ***abbabbCabbabb*** does not have an S-cover, *C* is not in any rightmost square so any collection of rightmost squares cannot satisfy the 3rd condition.

- The $(d, n-d)$ table setup allows us to limit the search for square-maximal strings to strings with an S-cover.
- Note that strings with an S-cover are necessarily free of singletons (i.e., letters with a single occurrence).
- Since generating a square requires just to generate its root, it basically doubles the length of strings that can be processed.
- There are some additional constraints on the strings based on the properties of the $(d, n-d)$ table.

Generating counter-examples

- We changed the paradigm – instead of generating possible candidates for maximality, we try to generate counter-examples to the d-step conjecture, i.e. (d, n) strings with strictly more than $n-d$ rightmost squares.
- This change of paradigm significantly reduces the search space.
- The reduction of the search space is in the form of several conditions a possible counter-example must satisfy.
- It allowed us to extend the viable length of strings that can be processed, and hence extend the range of the validity of the d-step conjecture.

- Problem: since the result of computation for given d and n is an empty set, i.e. no counter-example had been generated, the result cannot be verified; the certificate is the code.
- But it is the same problem as with computing the square-maximal strings; the certificate is the code, as the maximality cannot be easily independently verified.
- As we keep discovering new necessary conditions that a counter-example must satisfy, the ultimate goal is to show analytically that counter-examples cannot exist.

The first lemma shows that induction on $n-d$ is well-founded and possible:

Lemma

Let $\boldsymbol{x}$ be a singleton-free (d, n)-string, $1 \le d < n$ and let d_1 be the number of distinct symbols in a non-empty proper prefix (resp. suffix) of $\boldsymbol{x}$ of length n_1. Then, $n_1 - d_1 < n - d$.

So, for all work we are assuming that the d-step conjecture holds for every $n_1 - d_1 < n - d$, and we are trying to generate counter-examples for d and n.

Necessary conditions for a (d, n)-string $\boldsymbol{x}$ to be a counter-example:

- It must have a special S-cover (and hence be singleton free).
- It must satisfy several density conditions.

The S-cover $\{(a_i, b_i) \ : \ 1 \le i \le k\}$ is **special** if:

- $k > 1$, so the S-cover has at least 2 squares.
- There is a square $(1, b)$ so that $b < b_1$, hence $(1, b_1)$ is an FS-double square.
  Note that $a_1 = 1$ *and that* $(1, b_1)$ *and* $(1, b)$ *are unique squares, i.e. both rightmost and leftmost occurrences.*
- The last square $(a_k, b_k) = (a_k, n)$ is a unique square.
- There is a unique square (a, n) so that $a_k < a$, hence $(1, a_k{-}n{+}1)$ is an FS-double square in $\boldsymbol{y}$, where $\boldsymbol{y} = \boldsymbol{x}[n]\boldsymbol{x}[n{-}1]...\boldsymbol{x}[2]\boldsymbol{x}[1]$, the string $\boldsymbol{x}$ in reverse.
- $\forall i \in 1..k{-}1$, $a_{i+1} < b_i$ and the intersection $\boldsymbol{x}[a_{i+1}..b_i]$ must contain all characters common to $\boldsymbol{x}[1..b_i]$ and $\boldsymbol{x}[a_{i+1}..n]$.

The density conditions are expressed using two arrays: $\boldsymbol{B}(\boldsymbol{i},\boldsymbol{j})$ is defined as the number of rightmost squares that start and $\boldsymbol{E}(\boldsymbol{i},\boldsymbol{j})$ that end in the interval $i..j$.

Let $1 \leq k < n$, let d_2 be the number of distinct symbols of $\boldsymbol{x}[k+1 \mathinner{..} n]$, and let e be the number of distinct symbols occurring in both $\boldsymbol{x}[1 \mathinner{..} k]$ and $\boldsymbol{x}[k+1 \mathinner{..} n]$.

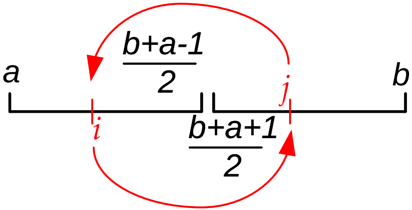
- $B(1,1) > 1$
- $B(1,2) > 2$
- $B(1,k) > k-d+d_2$
- $B(1,k)-E(1,k) > e$

In particular for binary strings:

- $B(1,k) > k$
- $B(1,k) - E(1,k) > 2$

An additional property that is not in the paper; let's define a symmetric and reflexive relation $\sim$ on $1..n$:

$i \sim j$ iff $i = j$ or there is a rightmost square (a, b) so that
$a \leq i \leq \frac{b+a-1}{2} < \frac{b+a+1}{2} \leq j \leq b$ and $i - a = j - \frac{b+a+1}{2}$ or
$a \leq j \leq \frac{b+a-1}{2} < \frac{b+a+1}{2} \leq i \leq b$ and $j - a = i - \frac{b+a+1}{2}$



Take the transitive closure of $\sim$. It is a relation of equivalence on $1..n$.

In simple terms, $i \sim j$ if $i = j$ or we can use successive rightmost squares to map i onto j.

Note that necessarily, if $i \sim j$, then $\boldsymbol{x}[i] = \boldsymbol{x}[j]$.

Necessary condition for the S-cover:

For any i in $1..a_{i+1}$ so that $\boldsymbol{x}[i]$ occurs in the intersection $\boldsymbol{x}[a_{i+1}..b_i]$, $i \sim j$ for some $j \in a_{i+1}..b_i$.

Conclusion

- We presented a set of necessary conditions for a counter-example for d-step conjecture for (d, n)-strings when the d-step conjecture is verified for all (d', n')-strings such that $n'-d' < n-d$.
- This allowed us to computationally verify the d-step conjecture for previously intractable lengths.
- The main goal is to discover more necessary conditions which would allow to prove the non-existence of counter-examples in an analytical way.

THANK YOU