Sorting suffixes of a text via its Lyndon Factorization

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Our goal

The goal is to introduce a new strategy for sorting the suffixes of a word w.

- The process of sorting the suffixes of a word plays a fundamental role in *Text Algorithms* with several applications in many areas of Computer Science and Bioinformatics.
- For instance, it is a fundamental step, in implicit or explicit way, for the construction of
  - the Suffix Array (SA): the array containing the starting positions of the suffixes of a word, sorted in lexicographic order;
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Sorting suffixes by Lyndon factorization

Our idea

Our strategy uses the *Lyndon factorization* and is based on a combinatorial property that allows to sort the suffixes of w ("global suffixes") by using the sorting of the suffixes inside blocks of consecutive Lyndon factors of the decomposition ("local suffixes").

Lyndon Words

- Two words $u, v \in \Sigma^*$ are conjugate, if $u = xy$ and $v = yx$ for some $x, y \in \Sigma^*$. Thus conjugate words are just cyclic shifts of one another.
- A word $w \in \Sigma^+$ is *primitive* if $w = u^h$ implies $w = u$ and $h = 1$.

Definition

A *Lyndon word* is a (primitive) word that is smaller in lexicographic order than all of its conjugates.

Example

- $u = mathematics$ is not a Lyndon word;
- $v = athematicsm$ is a Lyndon word.

There exist linear algorithms for the computation of the Lyndon word of a given word [*Duval*, 1983].

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Lyndon Factorization

Theorem (*Chen, Fox and Lyndon*: 1958)

Every word $w \in \Sigma^+$ has a unique factorization $w = L_1 \cdots L_k$ such that

$$L_1 \geq \cdots \geq L_k$$

is a non-increasing sequence of Lyndon words.

Let $w = abaaaabaaaaabaaaabaaaaaab$. The Lyndon factorization of w is

$$ab|aaaab|aaaaabaaaab|aaaaaab$$

Note that each L_i is strictly less than any of its proper conjugates/suffixes.

The Lyndon factorization of a given word can be computed

- in linear time [*Duval*, 1983];
- in parallel way [*Apostolico and Crochemore*, 1989] and [*Daykin, Iliopoulos and Smyth*, 1994];
- in external memory [*Roh, Crochemore, Iliopoulos and Par*, 2008].

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Local and Global suffixes

For each factor u of w, we denote by $first(u)$ and $last(u)$ the position of the first and the last symbol, respectively, of the factor u in w.

We denote by

- $suf_u(i) = w[i, last(u)]$ and we call it *local suffix* at the position i with respect to u.
- $suf(i) = w[i, n]$ and we call it *global suffix* of w at the position i.

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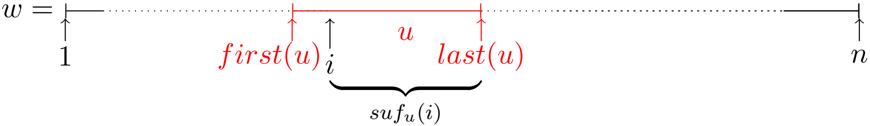

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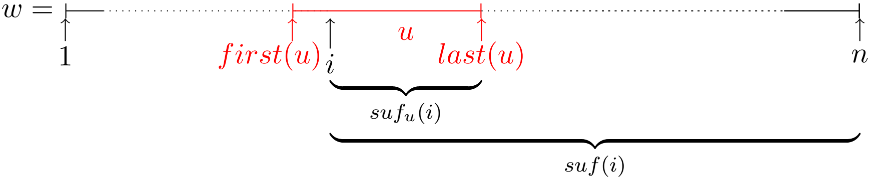

Compatible sorting

Definition

Let w be a word and let u be a factor of w. We say that the sorting of the *local* suffixes with respect to u is *compatible* with the sorting of the *global* suffixes of w if for all i, j with $first(u) \leq i < j \leq last(u)$,

$$suf_u(i) < suf_u(j) \iff suf(i) < suf(j).$$

In general, taken an arbitrary factor of a word w, the sorting of its suffixes is *not compatible* with the sorting of the suffixes of w, as the following example shows.

Example

Consider the word $w = abababb$ and its factor $u = ababa$.
Then $suf_u(1) = ababa > a = suf_u(5)$
whereas $suf(1) = abababb < abb = suf(5)$.
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Our result

Theorem

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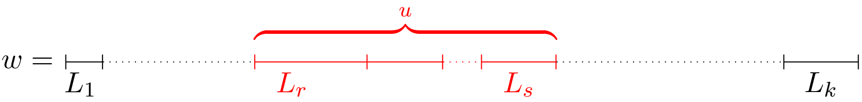

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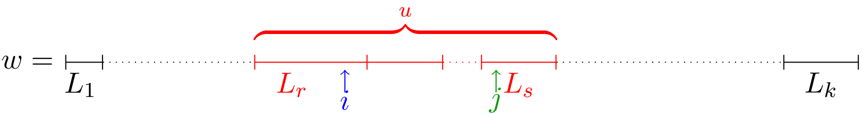

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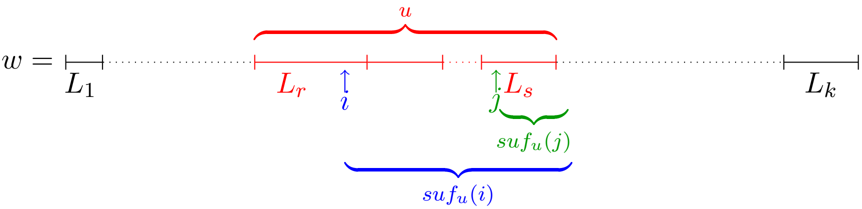

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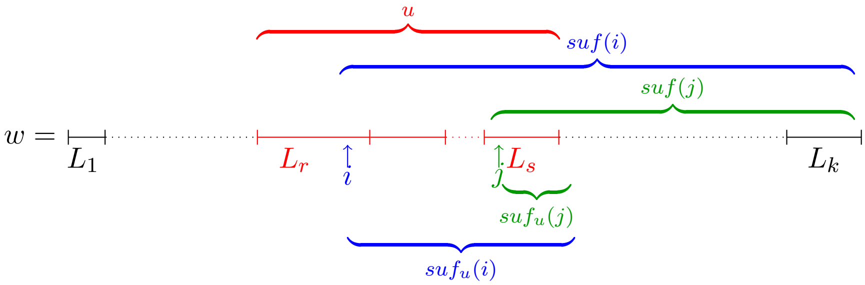
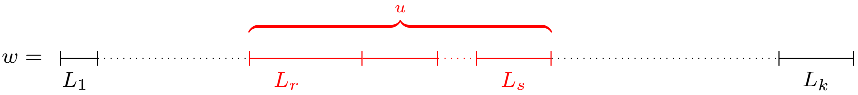


Easy case

The theorem is trivially true when the two suffixes start with two different Lyndon factors.

Suppose that

- i is the position of the first symbol of L_r
- j is the position of the first symbol of L_s
- u is the smallest factor containing both L_r and L_s: $L_rL_{r+1}\cdots L_s$.



Since $r < s$ and $L_1 \geq \cdots \geq L_r \geq \cdots \geq L_s \geq \cdots \geq L_k$. It is **easy** to verify that

- $L_rL_{r+1}\cdots L_s > L_s$
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We don't need to compare any symbol.

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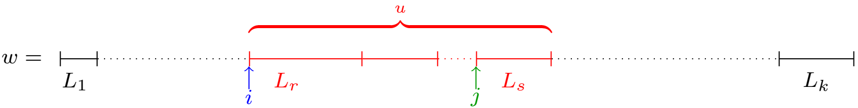

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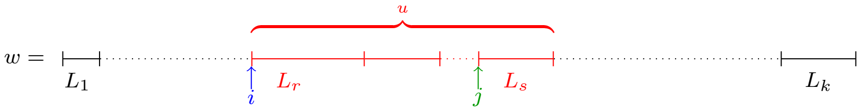

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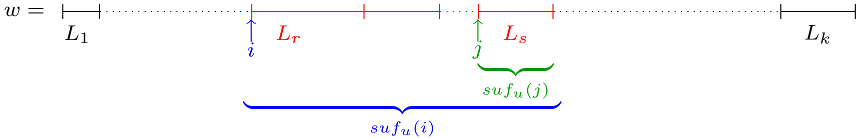

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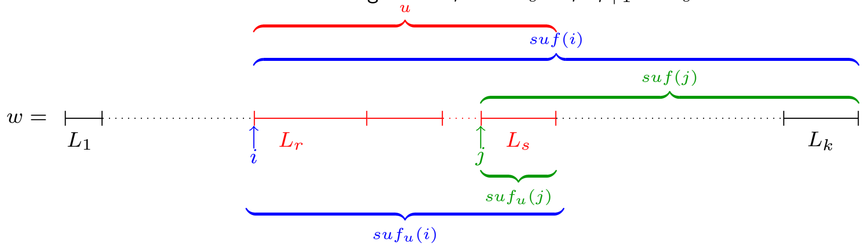

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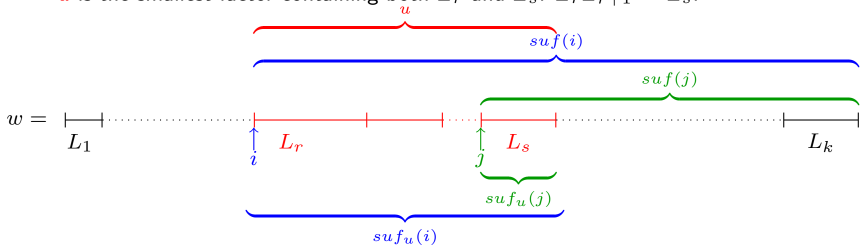

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Other cases

The theorem is true when the two suffixes of w start inside the same factor u of consecutive Lyndon words.

Suppose that

- i is a position inside L_r;
- j is a position inside L_s;
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$$suf(i) = \underbrace{L_r[i, last(L_r)] \; L_{r+1} \cdots L_s}_{suf_u(i)} \cdots L_k$$

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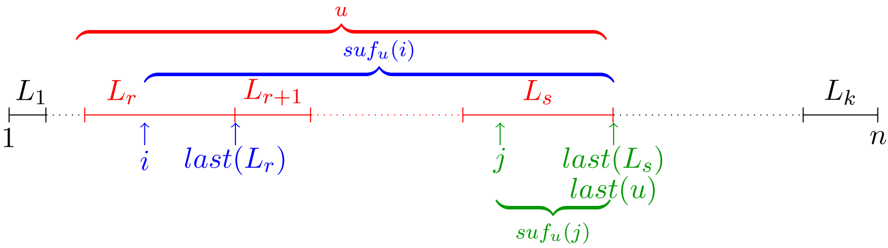

Possible cases:

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- There is not a different symbol inside $w[i, last(L_r)]$ and $w[j, last(L_s)]$:
  - $w[i, last(L_r)] = w[j, last(L_s)]$;
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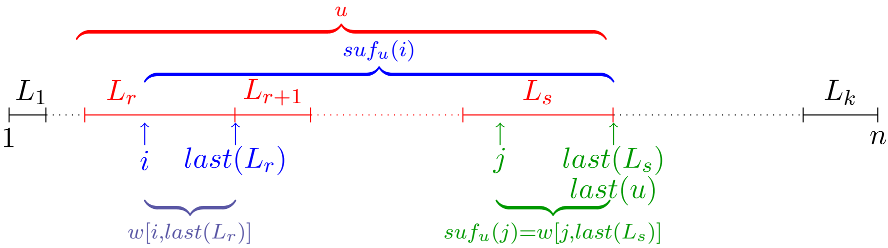

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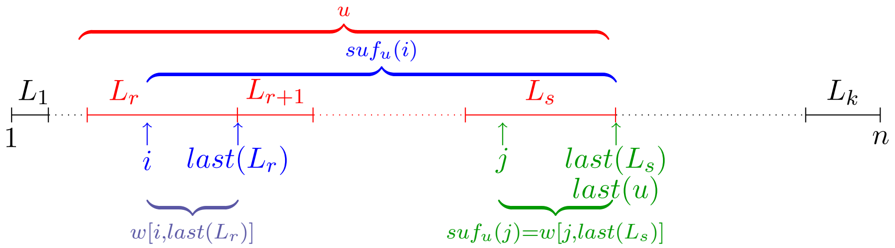
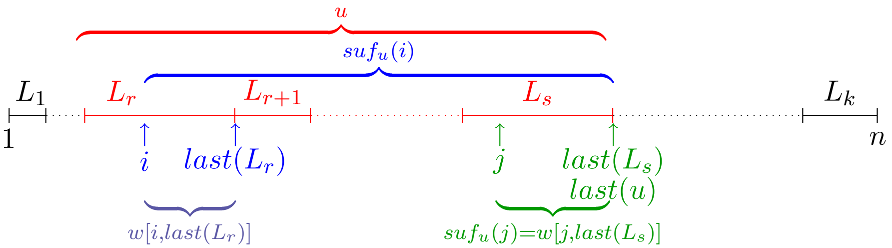


Possible cases:

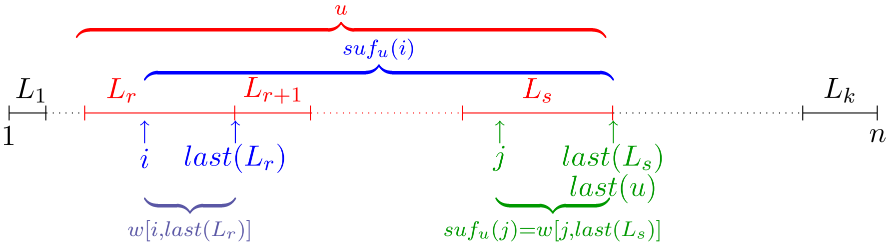
- There is a different symbol inside $w[i, last(L_r)]$ and $w[j, last(L_s)]$.
- There is not a different symbol inside $w[i, last(L_r)]$ and $w[j, last(L_s)]$:
  - $w[i, last(L_r)] = w[j, last(L_s)]$;
  - $w[j, last(L_s)]$ is a prefix of $w[i, last(L_r)]$;
  - $w[i, last(L_r)]$ is a prefix of $w[j, last(L_s)]$.

First case



- There is a different symbol inside $w[i, last(L_r)]$ and $w[j, last(L_s)]$.
- It is easy to verify that the order relation between the local and the global suffixes is the same!
- We need $lcp(i, j) + 1 \leq min(|w[i, last(L_r)]|, |w[j, last(L_s)]|)$ symbol comparisons, where $lcp(i, j)$ denotes the length of the longest common prefix between the suffixes $w[i, n]$ and $w[j, n]$.

First case



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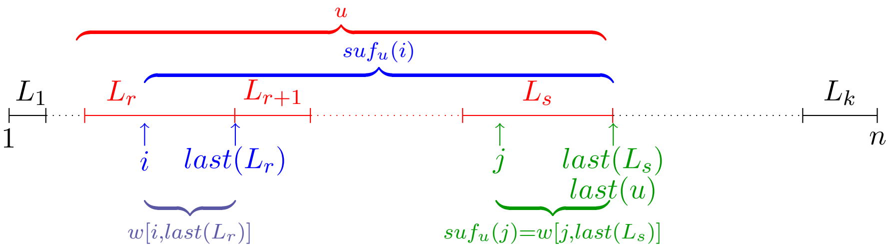

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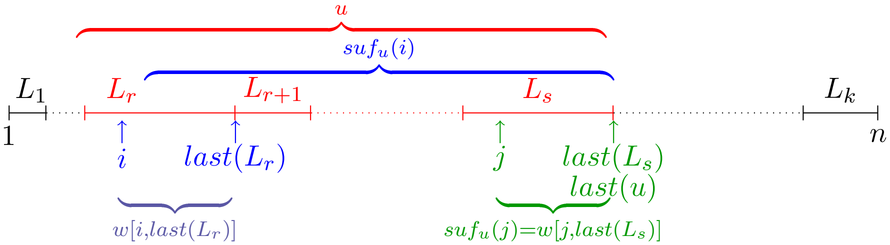
Second case: $w[i, last(L_r)] = w[j, last(L_s)]$



- Since $r < s$ and $L_1 \geq \cdots \geq L_r \geq \cdots \geq L_s \geq \cdots \geq L_k$. It is **easy** to verify that the order relation between the local and the global suffixes is the same! So we don't need to compare further symbols.
- We need $l(j) = |w[j, last(L_s)]| = |w[i, last(L_r)]|$ symbol comparisons.

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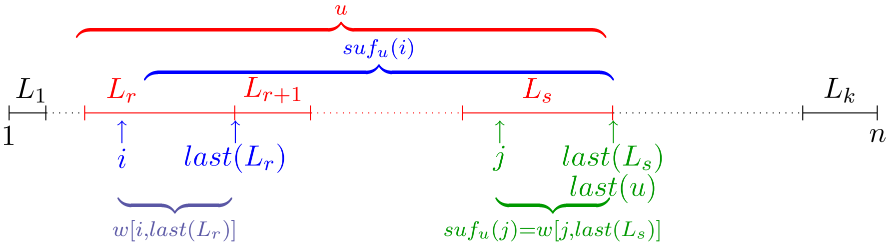

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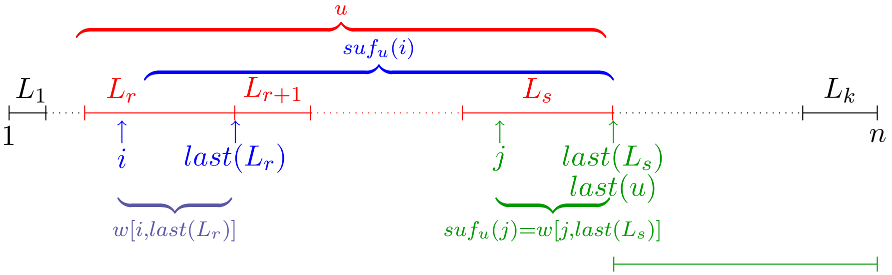

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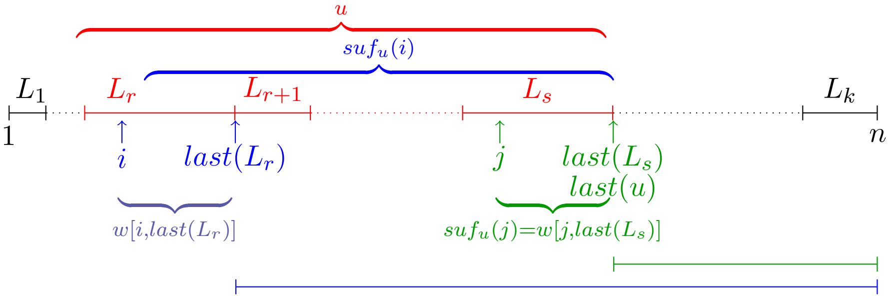

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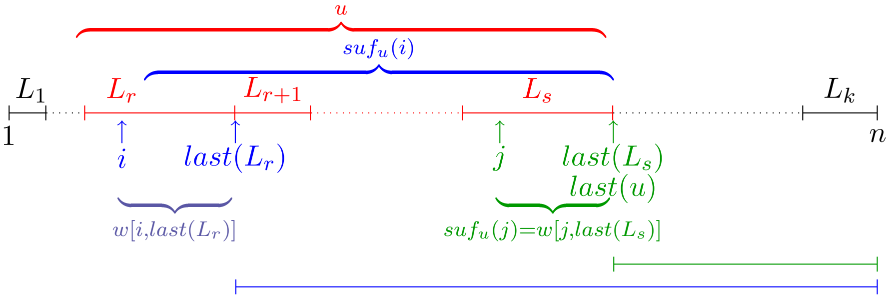

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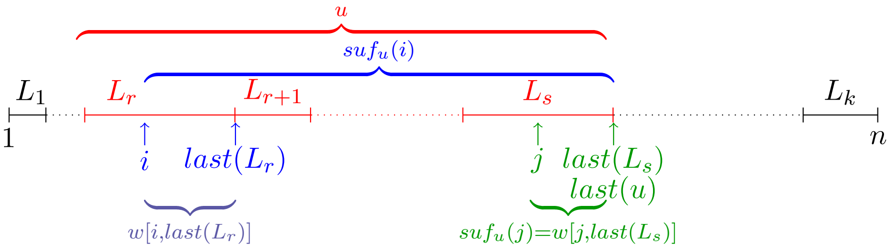
Second case: $w[j, last(L_s)]$ is a prefix of $w[i, last(L_r)]$



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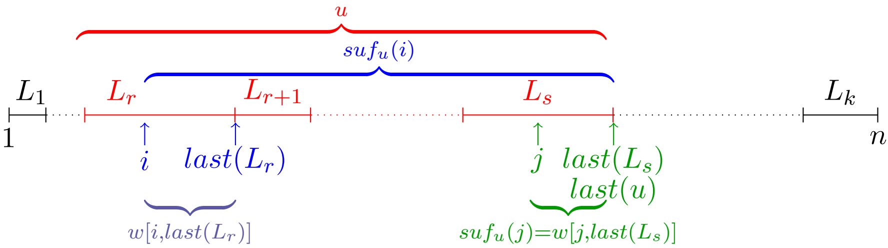

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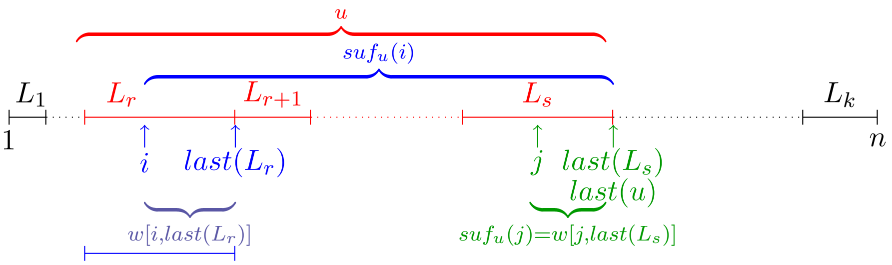

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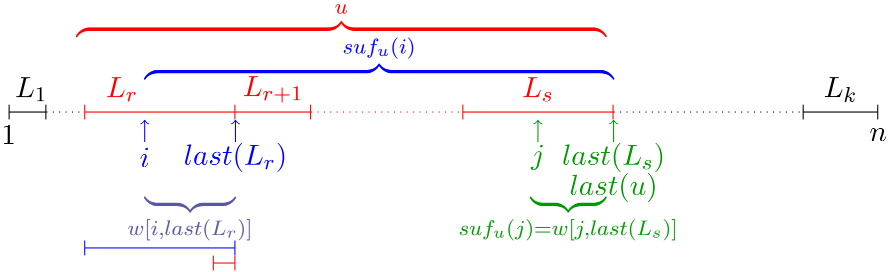

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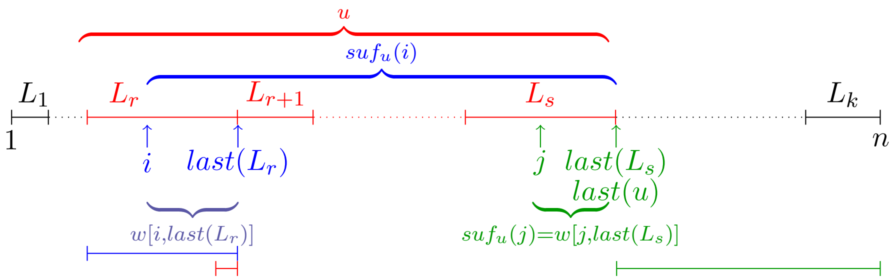

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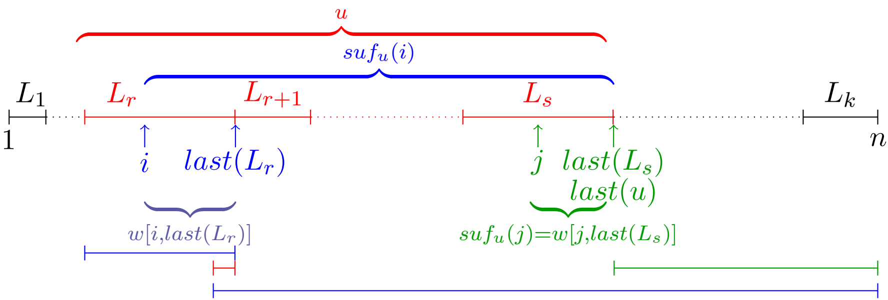

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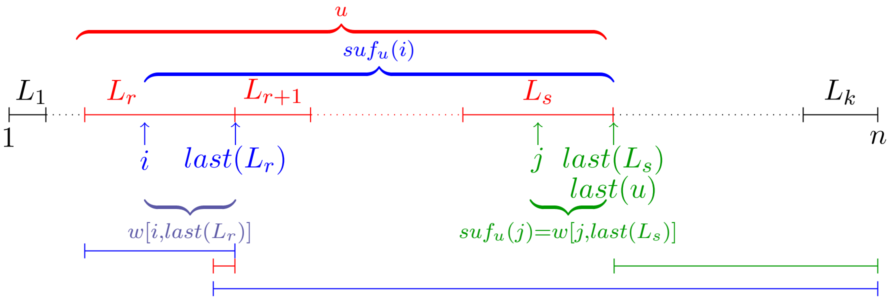

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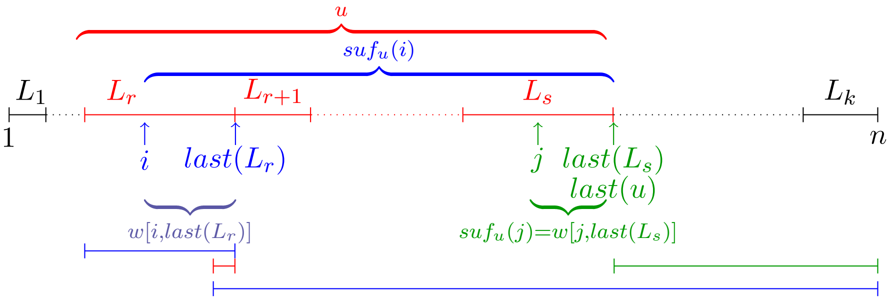

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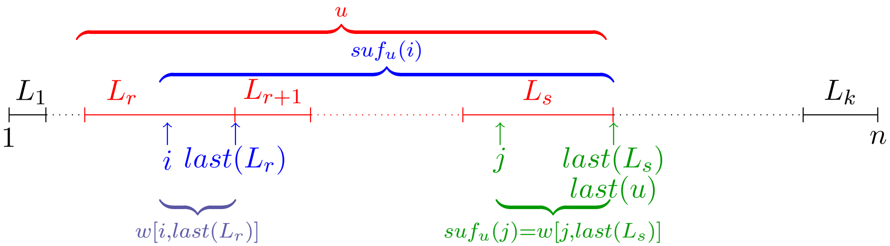

- In order to get the mutual order between $suf(i)$ and $suf(j)$, we need to compare at most $l(j) = |suf_u(j)|$ symbol comparisons.
- Consider $w[i, i + l(j) - 1]$ and $w[j, j + l(j) - 1] = suf_u(j)$.
  - There is a mismatch, then we need $lcp(i, j) + 1 \leq l(j)$ symbol comparisons.
  - There is not a mismatch, then we use the property of the Lyndon factorization: $L_{s+1} \cdots L_k$ is smaller than any suffix of

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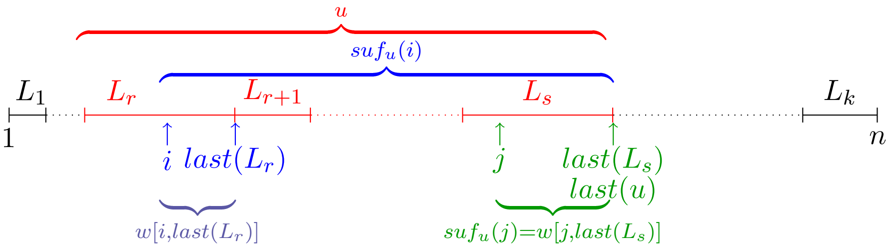

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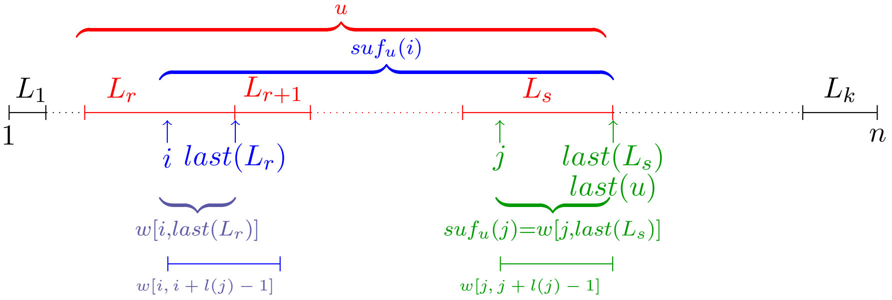

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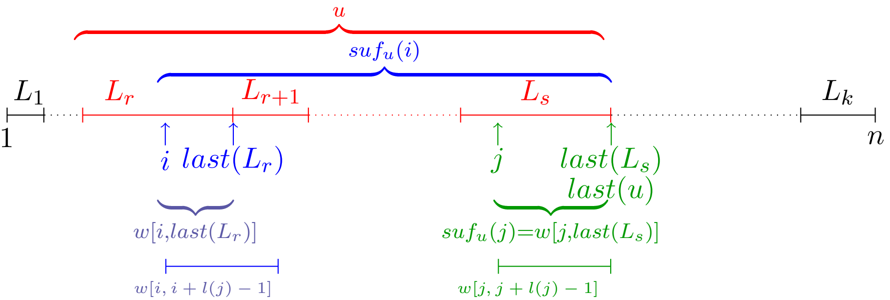

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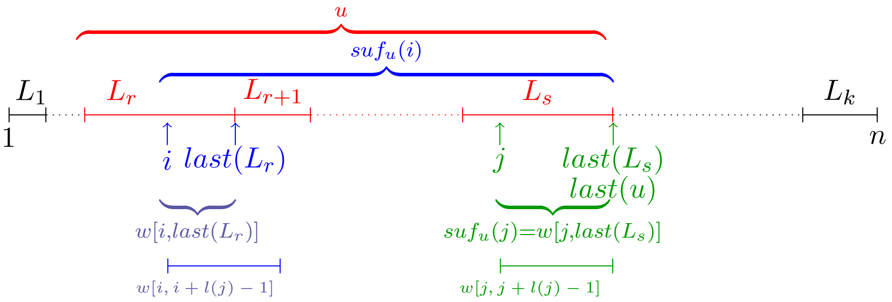

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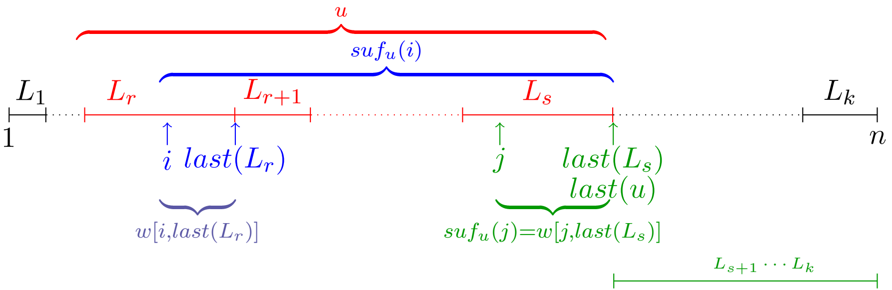

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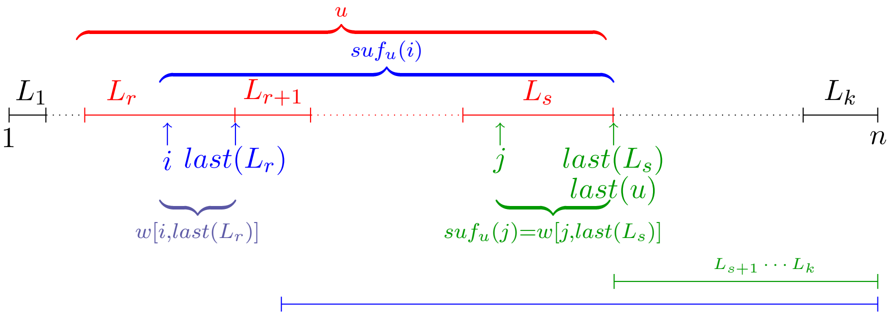

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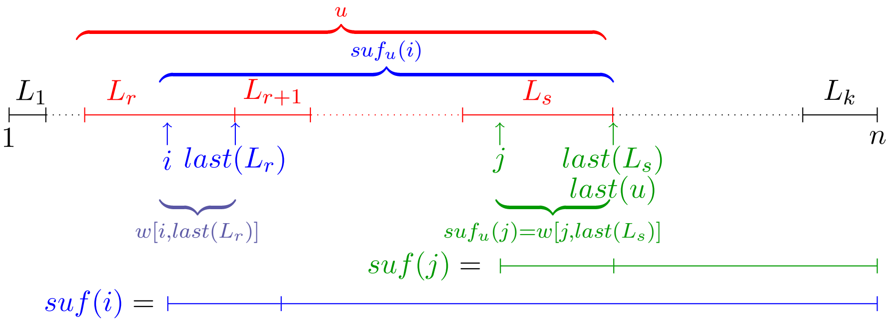

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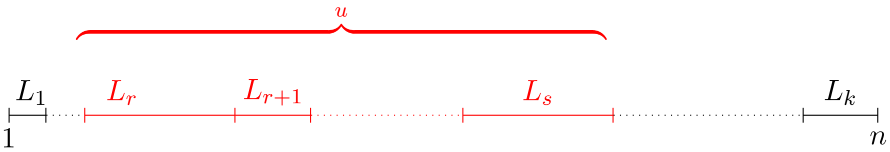
How many symbol comparisons?



- In order to get the mutual order between $suf(i)$ and $suf(j)$ it is sufficient to execute at most $l(j) = |suf_u(j)|$ symbol comparisons.
- Note that $l(j)$, as shown by the following example, can be smaller than $lcp(i, j) + 1$.

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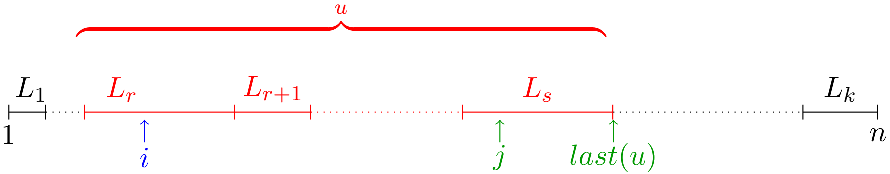

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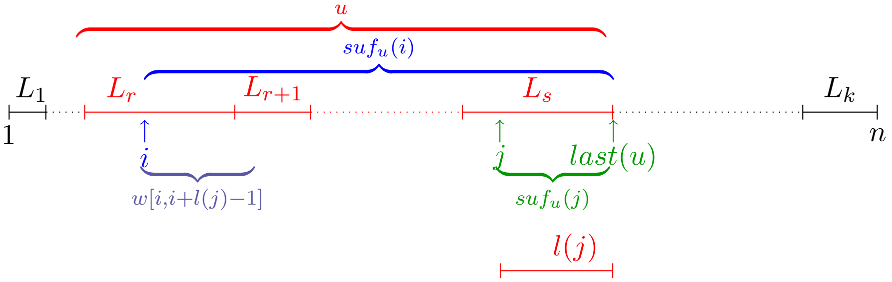

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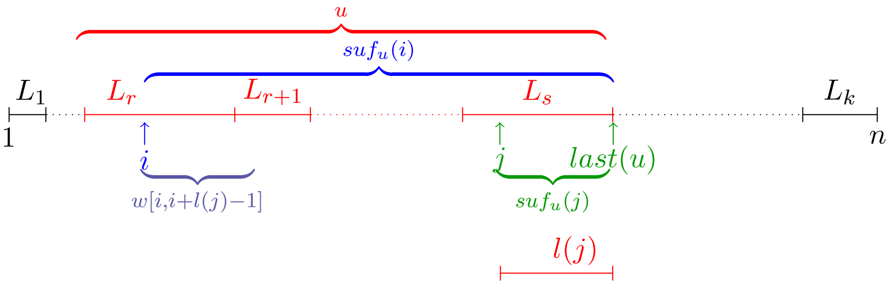

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Example

Let $w = abaaaabaaaaabaaaabaaaaaab$. Its Lyndon factorization is $ab|aaaab|aaaaabaaaab|aaaaaab$. Let $u = ab|aaaab|aaaaabaaaab|$.

$$\begin{array}{cccccccccccccccccccccccccccc}
 & & \overset{i=2}{\downarrow} & & & & & & & & & & & & & \overset{j=13}{\downarrow} & & & & & & & & & & & \\
 & 1 & \mathbf{2} & & 3 & 4 & 5 & 6 & 7 & & 8 & 9 & 10 & 11 & 12 & \mathbf{13} & 14 & 15 & 16 & 17 & 18 & & 19 & 20 & 21 & 22 & 23 & 24 & 25 & \\
w = & a & \mathbf{b} & | & a & a & a & a & b & | & a & a & a & a & a & \mathbf{b} & a & a & a & a & b & | & a & a & a & a & a & a & b & |
\end{array}$$

Consider the following suffixes:

$$\begin{array}{rl}
 & \overset{2}{\downarrow} \\
suf(2) = & b\ a\ a\ a\ a\ b\ a\ a\ a\ a\ a\ b\ a\ a\ a\ a\ b\ a\ a\ a\ a\ a\ a\ b \\
suf(13) = & b\ a\ a\ a\ a\ b\ a\ a\ a\ a\ a\ a\ b \\
 & \underset{13}{\uparrow}
\end{array}$$

We have $lcp(2, 13) = 11$ and $l(13) = 6$.

We need only 6 symbol comparisons, indeed for Lyndon properties $w[8, 25] > w[19, 25] \Rightarrow suf(2) > suf(13)$.

Example

Let $w = abaaaabaaaaabaaaabaaaaaab$. Its Lyndon factorization is $ab|aaaab|aaaaabaaaab|aaaaaab$. Let $u = ab|aaaab|aaaaabaaaab|$.

$$\begin{array}{cccccccccccccccccccccccccccccc}
 & & i=2 & & & & & & & & & & & & & j=13 & & & & & & & & & & & & & & \\
 & & \downarrow & & & & & & & & & & & & & \downarrow & & & & & & & & & & & & & & \\
 & 1 & \mathbf{2} & & 3 & 4 & 5 & 6 & 7 & & 8 & 9 & 10 & 11 & 12 & \mathbf{13} & 14 & 15 & 16 & 17 & 18 & & 19 & 20 & 21 & 22 & 23 & 24 & 25 & \\
w = & a & \mathbf{b} & | & a & a & a & a & b & | & a & a & a & a & a & \mathbf{b} & a & a & a & a & b & | & a & a & a & a & a & a & b & | \\
 & & & & & & & & & & & & & & \uparrow & & & & & & & & & & & & \uparrow & & & \\
 & & & & & & & & & & & & & & 2+lcp(2,13)-1 & & & & & & & & & & & & 13+lcp(2,13)-1 & & &
\end{array}$$

Consider the following suffixes:

$$\begin{array}{cccccccccccccc}
 & 2 & & & & & & & & & & & 2+lcp(2,13) & \\
 & \downarrow & & & & & & & & & & & \downarrow & \\
suf(2) = & b & a & a & a & a & b & a & a & a & a & a & b & a \quad a\,a\,a\,b\,a\,a\,a\,a\,a\,a\,b \\
suf(13) = & b & a & a & a & a & b & a & a & a & a & a & a & b \\
 & \uparrow & & & & & & & & & & & \uparrow & \\
 & 13 & & & & & & & & & & & 13+lcp(2,13) &
\end{array}$$

We have $lcp(2,13) = 11$ and $l(13) = 6$.

We need only 6 symbol comparisons, indeed for Lyndon properties $w[8,25] > w[19,25] \Rightarrow suf(2) > suf(13)$.

Example

Let $w = abaaaabaaaaabaaaabaaaaaab$. Its Lyndon factorization is $ab|aaaab|aaaaabaaaab|aaaaaab$. Let $u = ab|aaaab|aaaaabaaaab|$.

$$\begin{array}{cccccccccccccccccccccccccccccc}
 & & i=2 & & & & & & & & & & & & & j=13 & & & & & last(u)=18 & & & & & & & & \\
 & & \downarrow & & & & & & & & & & & & & \downarrow & & & & & \downarrow & & & & & & & & \\
 & 1 & \mathbf{2} & & 3 & 4 & 5 & 6 & 7 & & 8 & 9 & 10 & 11 & 12 & \mathbf{13} & 14 & 15 & 16 & 17 & 18 & & 19 & 20 & 21 & 22 & 23 & 24 & 25 & \\
w = & a & \mathbf{b} & | & a & a & a & a & b & | & a & a & a & a & a & \mathbf{b} & a & a & a & a & b & | & a & a & a & a & a & a & b & | \\
 & & & & & & & & \uparrow & & & & & & & & & & & & \uparrow & & & & & & & & \\
 & & & & & & & & 2+l(13)-1 & & & & & & & & & & & & 13+l(13)-1 & & & & & & & &
\end{array}$$

Consider the following suffixes:

$$\begin{array}{lcccccccccccccc}
 & 2 & & & & & 2+l(13) & & & & & & 2+lcp(2,13) & & \\
 & \downarrow & & & & & \downarrow & & & & & & \downarrow & & \\
suf(2) = & b & a & a & a & a & b & a & a & a & a & a & b & a & a\,a\,a\,b\,a\,a\,a\,a\,a\,a\,b \\
suf(13) = & b & a & a & a & a & b & a & a & a & a & a & a & b & \\
 & \uparrow & & & & & \uparrow & & & & & & \uparrow & & \\
 & 13 & & & & & 13+l(13) & & & & & & 13+lcp(2,13) & &
\end{array}$$

We have $lcp(2,13) = 11$ and $l(13) = 6$.

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Let $w = abaaaabaaaaabaaaabaaaaaab$. Its Lyndon factorization is $ab|aaaab|aaaaabaaaab|aaaaaab$. Let $u = ab|aaaab|aaaaabaaaab|$.

$$
\begin{array}{ccccccccccccccccccccccccccccccc}
 & & i=2 & & & & & & & & & & & & & j=13 & & & & & last(u)=18 & & & & & & & & & \\
 & & \downarrow & & & & & & & & & & & & & \downarrow & & & & & \downarrow & & & & & & & & & \\
 & 1 & \mathbf{2} & & 3 & 4 & 5 & 6 & 7 & & 8 & 9 & 10 & 11 & 12 & \mathbf{13} & 14 & 15 & 16 & 17 & 18 & & 19 & 20 & 21 & 22 & 23 & 24 & 25 & \\
w = & a & \mathbf{b} & | & a & a & a & a & b & | & a & a & a & a & a & \mathbf{b} & a & a & a & a & b & | & a & a & a & a & a & a & b & | \\
 & & & & & & & & \uparrow & & & & & & & & & & & & \uparrow & & & & & & & & & \\
 & & & & & & & & 2+l(13)-1 & & & & & & & & & & & & 13+l(13)-1 & & & & & & & & &
\end{array}
$$

Consider the following suffixes:

$$
\begin{array}{rcccccccccccccccccccccccccc}
 & 2 & & & & & 2+l(13) & & & & & & 2+lcp(2,13) & & & & & & & & & & & & \\
 & \downarrow & & & & & \downarrow & & & & & & \downarrow & & & & & & & & & & & & \\
suf(2) = & b & a & a & a & a & b & a & a & a & a & a & b & a & a & a & a & b & a & a & a & a & a & a & b \\
suf(13) = & b & a & a & a & a & b & a & a & a & a & a & a & b & & & & & & & & & & & \\
 & \uparrow & & & & & \uparrow & & & & & & \uparrow & & & & & & & & & & & & \\
 & 13 & & & & & 13+l(13) & & & & & & 13+lcp(2,13) & & & & & & & & & & & &
\end{array}
$$

We have $lcp(2,13) = 11$ and $l(13) = 6$.

We need only 6 symbol comparisons, indeed for Lyndon properties $w[8,25] > w[19,25] \Rightarrow suf(2) > suf(13)$.

Our strategy for sorting all suffixes

Let $w = L_1 L_2 \cdots L_l L_{l+1} \cdots L_k$. We propose an algorithm that is based on the following

Proposition

Let $sort(L_1 L_2 \cdots L_l)$ and $sort(L_{l+1} L_{l+2} \cdots L_k)$ denote the sorted lists of the suffixes of $L_1 L_2 \cdots L_l$ and the suffixes $L_{l+1} L_{l+2} \cdots L_k$, respectively. Then
$sort(L_1 L_2 \cdots L_k) = merge(sort(L_1 L_2 \cdots L_l), sort(L_{l+1} L_{l+2} \cdots L_k))$.

- The sorted list of the global suffixes of w can be obtained by merging the sorted lists of the local suffixes inside $L_1 L_2 \cdots L_l$ and $L_{l+1} L_{l+2} \cdots L_k$.
- Note that the mutual order of the local suffixes is preserved after the merge operation.

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Our algorithm

This proposition suggests a possible strategy for sorting the list of the suffixes of some word w:

- find the Lyndon decomposition of w: $L_1 L_2 \cdots L_k$;
- find the sorted list of the suffixes of L_1 and, separately, the sorted list of the suffixes of L_2;
- merge the sorted lists in order to obtain the sorted list of the suffixes of $L_1 L_2$;
- find the sorted list of the suffixes of L_3 and merge it to the previous sorted list;
- repeat until all the Lyndon factors are processed;

One can use this strategy for computing the suffix array and for constructing the Burrows-Wheeler Transform.

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Compute the BWT and the SA

- the Suffix Array (SA): the array containing the starting positions of the suffixes of a word, sorted in lexicographic order;
- the Burrows-Wheeler Transform (BWT): the array containing a permutation of the symbols of a word according to the sorting of its suffixes.

Let $w = aabcabbaabaabdabbaaabbdc$. Its Lyndon factorization is $aabcabb|aabaabdabb|aaabbdc$.

$$w\$ = \overbrace{\vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!\dashv}^{L_1 = aabcabb} \overbrace{\vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!\dashv}^{L_2 = aabaabdabb} \overbrace{\vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!\dashv}^{L_3 = aaabbdc} \overbrace{\vdash\!\!\!-\!\!\!\dashv}^{L_4 = \$}$$

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$$w\$ = \underbrace{\vphantom{|}}_{}\; L_1 = aabcabb \quad L_2 = aabaabdabb \quad L_3 = aaabbdc \quad L_4 = \$$$

Consider: $L_1\$ = aabcabb\$$

Compute the BWT and the SA

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$w\$ =$ $L_1 = aabcabb$ $L_2 = aabaabdabb$ $L_3 = aaabbdc$ $L_4 = \$$

Consider: $L_1\$ = aabcabb\$$

Compute the $BWT(L_1\$)$ and $SA(L_1\$)$:

$L_1\$$

SA	BWT	Sorted Suffixes
8	$\underline{b}$	$\$$
1	$\$$	$aabcabb\$$
5	c	$abb\$$
2	a	$abcabb\$$
7	b	$b\$$
6	a	$bb\$$
3	a	$bcabb\$$
4	b	$cabb\$$

Compute the BWT and the SA

$$w = \underbrace{L_1 = aabcabb}\ \underbrace{\color{red}{L_2 = aabaabdabb}}\ \underbrace{L_3 = aaabbdc}$$

$$L_1\$ = aabcabb\$$$

$L_1\$$

	SA	BWT	Sorted Suffixes
	8	$\underline{b}$	\$
	1	\$	$aabcabb\$$
	5	c	$abb\$$
	2	a	$abcabb\$$
	7	b	$b\$$
	6	a	$bb\$$
	3	a	$bcabb\$$
	4	b	$cabb\$$

Compute the BWT and the SA

$$w = \underbrace{L_1 = aabcabb}\ \underbrace{L_2 = aabaabdabb}\ \underbrace{L_3 = aaabbdc}$$

$$\underbrace{L_1\$ = aabcabb\$}$$

Consider: $\underbrace{L_2\$ = aabaabdabb\$}$

Note that $|L_1| = j_1 = 7$. Compute the $BWT(L_2\$)$ and $SA(L_2\$)$.

$L_1\$$

SA	BWT	Sorted Suffixes
8	$\underline{b}$	$\$$
1	$\$$	$aabcabb\$$
5	c	$abb\$$
2	a	$abcabb\$$
7	b	$b\$$
6	a	$bb\$$
3	a	$bcabb\$$
4	b	$cabb\$$

$L_2\$$

SA	BWT	Sorted Suffixes
$11+7=18$	b	$\$$
$1+7=8$	$\$$	$\underline{aabaabdabb\$}$
$4+7=11$	b	$aabdabb\$$
$2+7=9$	a	$abaabdabb\$$
$8+7=15$	d	$abb\$$
$5+7=12$	a	$abdabb\$$
$10+7=17$	b	$b\$$
$3+7=10$	a	$baabdabb\$$
$9+7=16$	a	$bb\$$
$6+7=13$	a	$bdabb\$$
$7+7=14$	b	$dabb\$$

Compute the BWT and the SA

$L_1\$$

	SA	BWT	Sorted Suffixes
	8	$\underline{b}$	$\$$
	1	$\$$	$aabcabb\$$
	5	c	$abb\$$
	2	a	$abcabb\$$
	7	b	$b\$$
	6	a	$bb\$$
	3	a	$bcabb\$$
	4	b	$cabb\$$

$L_2\$$

G	SA	BWT	Sorted Suffixes
0	$11+7=18$	b	$\$$
0	$1+7=8$	$\$$	$\underline{aabaabdabb\$}$
2	$4+7=11$	b	$aabdabb\$$
2	$2+7=9$	a	$abaabdabb\$$
2	$8+7=15$	d	$abb\$$
4	$5+7=12$	a	$abdabb\$$
4	$10+7=17$	b	$b\$$
5	$3+7=10$	a	$baabdabb\$$
5	$9+7=16$	a	$bb\$$
7	$6+7=13$	a	$bdabb\$$
8	$7+7=14$	b	$dabb\$$

merge $\Rightarrow$

$L_1 L_2\$$

SA	BWT	Sorted Suffixes
18	b	$\$$
$\underline{8}$	$\underline{b}$	$\underline{aabaabdabb\$}$
1	$\$$	$aabcabbaabaabdabb\$$
11	b	$aabdabb\$$
9	a	$abaabdabb\$$
15	d	$abb\$$
5	c	$abbaabaabdabb\$$
2	a	$abcabbaabaabdabb\$$
12	a	$abdabb\$$
17	b	$b\$$
7	b	$baabaabdabb\$$
10	a	$baabdabb\$$
16	a	$bb\$$
6	a	$bbaabaabdabb\$$
3	a	$bcabbaabaabdabb\$$
13	a	$bdabb\$$
4	b	$cabbaabaabdabb\$$
14	b	$dabb\$$

Compute the BWT and the SA

$w =$ $L_1 = aabcabb$ $\quad$ $L_2 = aabaabdabb$ $\quad$ $L_3 = aaabbdc$

$L_1L_2\$ = aabcabbaabaabdabb\$$

By merging the sorted list of the suffixes of $L_1L_2\$$ and of $L_3\$$, we obtain the SA/BWT of $w\$ = L_1L_2L_3\$$.

Compute the BWT and the SA

$w = \overbrace{}^{L_1 = aabcabb}\ \overbrace{}^{L_2 = aabaabdabb}\ \overbrace{}^{L_3 = aaabbdc}$

$L_1 L_2\$ = aabcabbaabaabdabb\$$

Consider: $L_3\$ = aaabbdc\$$

Compute the $BWT(L_3\$)$ and $SA(L_3\$)$.

SA	$L_3\$$ BWT	Sorted Suffixes
$17+8=25$	c	$\$$
$17+1=18$	$\$$	$aaabbdc\$$
$17+2=19$	a	$aabbdc\$$
$17+3=20$	a	$abbdc\$$
$17+4=21$	a	$bbdc\$$
$17+5=22$	b	$bdc\$$
$17+7=24$	d	$c\$$
$17+6=23$	b	$dc\$$

By merging the sorted list of the suffixes of $L_1L_2\$$ and of $L_3\$$, we obtain the SA/BWT of $w\$ = L_1L_2L_3\$$.

Compute the BWT and the SA

$w = \underbrace{\hspace{3em}}_{L_1 = aabcabb}\underbrace{\hspace{3em}}_{L_2 = aabaabdabb}\underbrace{\hspace{3em}}_{L_3 = aaabbdc}$

$L_1 L_2\$ = aabcabbaabaabdabb\$$

Consider: $L_3\$ = aaabbdc\$$

Compute the $BWT(L_3\$)$ and $SA(L_3\$)$.

SA	$L_3\$$ BWT	Sorted Suffixes
$17+8=25$	c	$\$$
$17+1=18$	$\$$	$aaabbdc\$$
$17+2=19$	a	$aabbdc\$$
$17+3=20$	a	$abbdc\$$
$17+4=21$	a	$bbdc\$$
$17+5=22$	b	$bdc\$$
$17+7=24$	d	$c\$$
$17+6=23$	b	$dc\$$

By merging the sorted list of the suffixes of $L_1L_2\$$ and of $L_3\$$, we obtain the SA/BWT of $w\$ = L_1L_2L_3\$$.

Further work: Parallel sorting

- The word could be partitioned into several sequences of consecutive blocks of Lyndon words, and the sorting algorithm can be applied *in parallel* to each of those sequences. Then one should merge the sorted lists.
- Furthermore, also the Lyndon factorization can be performed in parallel, as shown in [*Apostolico and Crochemore*, 1989] and [*Daykin, Iliopoulos and Smyth*, 1994].

Further work

One can compute the BWT without the SA by using our strategy and the strategies already used in the following papers:

- *Hon, Lam, Sadakane, Sung and Yiu*, 2007;
- *Ferragina, Gagie and Manzini*, 2010 and 2012;
- *Bauer, Cox and R.*, 2011 and 2013;
- *Crochemore, Grossi, Kärkkäinen and Landau*, 2013.

In this way, one could obtain algorithms that work:

- in external memory;
- in place.

One could use efficient dynamic data structures for the rank and insert operations, for instance by using Navarro and Nekrich's recent results on optimal representations of dynamic sequences.

Further work: linear algorithm

Does there exist a linear algorithm that uses the Lyndon Factorization in order to sort (implicity or explicity) the suffixes?

Open problem!

Thank you for your attention!