The Sum of Exponents of Maximal Repetitions in Standard Sturmian Words

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Sturmian words

Sturmian words are infinite words that has exactly $k+1$ factors of length k for each $n \geq 0$.

Example:

$a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,a\,b\ldots$

1. $a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,a\,b\ldots$
2. $a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,a\,b\ldots$
3. $a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,a\,b\ldots$
4. $a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,a\,b\ldots$

Standard word

- Alphabet: $\Sigma = \{a, b\}$
- Directive sequence: $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$, where $\gamma_0 \geq 0$ and $\gamma_1, \ldots, \gamma_n > 0$
- Recurrence:
  - $x_{-1} = b$
  - $x_0 = a$
  - $x_k = (x_{k-1})^{\gamma_{k-1}} \cdot x_{k-2}$
- $\mathrm{Sw}(\gamma_0, \gamma_1, \ldots, \gamma_n) = x_{n+1}$

Standard word example

- $\gamma = (1, 2, 1, 3, 1)$

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Standard word example

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- $x_{-1} = b$
- $x_0 = a$
- $x_1 = (x_0)^1 \cdot x_{-1} = a \cdot b$

Standard word example

- $\gamma = (1, 2, 1, 3, 1)$

- $x_{-1} = b$
- $x_0 = a$
- $x_1 = (x_0)^1 \cdot x_{-1} = a \cdot b$
- $x_2 = (x_1)^2 \cdot x_0 = ab \cdot ab \cdot a$

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- $x_3 = (x_2)^1 \cdot x_1 = ababa \cdot ab$
- $x_4 = (x_3)^3 \cdot x_2 = ababaab \cdot ababaab \cdot ababaab \cdot ababa$

Standard word example

- $\gamma = (1, 2, 1, 3, 1)$

- $x_{-1} = b$
- $x_0 = a$
- $x_1 = (x_0)^1 \cdot x_{-1} = a \cdot b$
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- $x_4 = (x_3)^3 \cdot x_2 = ababaab \cdot ababaab \cdot ababaab \cdot ababa$
- $x_5 = (x_4)^1 \cdot x_3 = ababaababaababaababaababa \cdot ababaab$

Standard word example

- $\gamma \;=\; (1,2,1,3,1)$

- $x_{-1} \;=\; b$
- $x_0 = a$
- $x_1 \;=\; (x_0)^1 \cdot x_{-1} \;=\; a \cdot b$
- $x_2 \;=\; (x_1)^2 \cdot x_0 \;=\; ab \cdot ab \cdot a$
- $x_3 \;=\; (x_2)^1 \cdot x_1 \;=\; ababa \cdot ab$
- $x_4 \;=\; (x_3)^3 \cdot x_2 \;=\; ababaab \cdot ababaab \cdot ababaab \cdot ababa$
- $x_5 \;=\; (x_4)^1 \cdot x_3 \;=\; ababaababaababaababaababa \cdot ababaab$

- $\mathrm{Sw}(1,2,1,3,1) \;=\; ababaababaababaababaababaababaab$

Fibonacci words

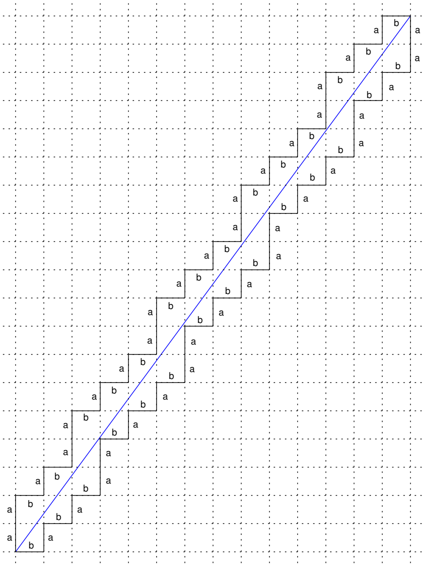
- $F_{-1} \;=\; b$
- $F_0 \;=\; a$
- $\vdots$
- $F_{n+1} \;=\; F_n \cdot F_{n-1}$

Fibonacci words

- $F_{-1} \;=\; b$
- $F_0 \;=\; a$
- $\vdots$
- $F_{n+1} \;=\; F_n \cdot F_{n-1}$

Observation

$$F_n \;=\; \mathrm{Sw}(\underbrace{1, 1, \ldots, 1, 1}_{n})$$



Lower Christoffel word:

*babab**a**ababaabababaabababaabababaababaa*

Upper Christoffel word:

aababaabababaabababaabababaababab

Standard words:

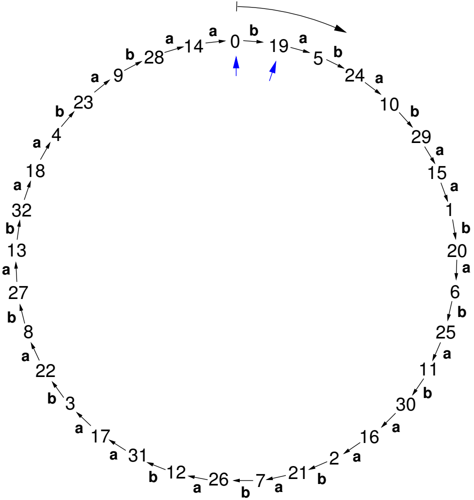
ababaabababaabababaabababaababaab

ababaabababaabababaabababaabababa

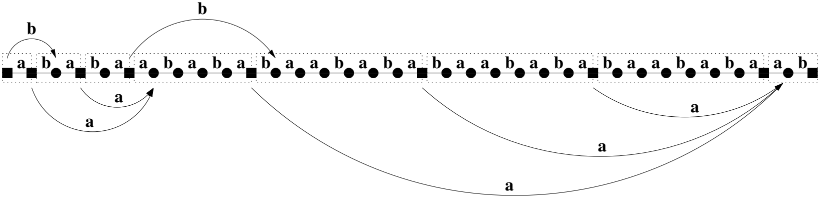
Proposition

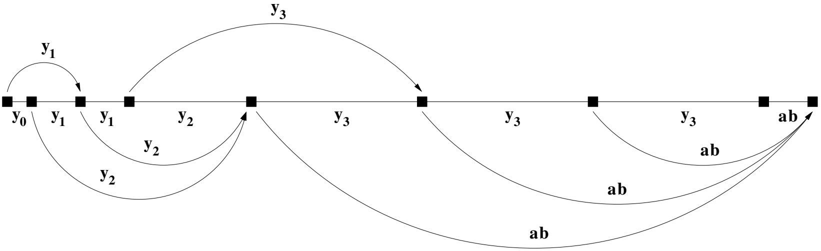
For fixed relatively prime $p > q > 0$ there exists a unique word x such that:

- $|x|_a \; = \; p - 1$
- $|x|_b \; = \; q - 1$
- $b \cdot x \cdot a$ – lower Christoffel word
- $a \cdot x \cdot b$ – upper Christoffel word
- $x \cdot ab$ – standard word
- $x \cdot ba$ – standard word



b a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a





Notation: $y_i \;=\; (x_i)^R$

Run (maximal repetition)

$$\underbrace{\mathrm{ababaab}}\underbrace{\mathrm{ababaab}}\underbrace{\mathrm{ababaab}}\underbrace{\mathrm{ababaab}}\underbrace{\mathrm{aba}}\mathrm{ab}$$

Details:

- starting position
- period (generator)
- exponent
- remainder
- length

Run (maximal repetition)

$\underbrace{ababa}\underbrace{ababa}\underbrace{ababa}\underbrace{ababa}\underbrace{aba}$ab

Details:

- starting position
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Run (maximal repetition)

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Details:

- starting position
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Details:

- starting position
- period (generator)
- exponent
- remainder
- length

Run (maximal repetition)

$$\underbrace{\text{ababaab}}\underbrace{\text{ababaab}}\underbrace{\text{ababaab}}\underbrace{\text{ababaab}}\underbrace{\color{blue}\text{aba}}\text{ab}$$

Details:

- starting position
- period (generator)
- exponent
- remainder
- length

Run (maximal repetition)

$$\underbrace{ababaab}\underbrace{ababaab}\underbrace{ababaab}\underbrace{ababaab}\underbrace{aba}ab$$

Details:

- starting position
- period (generator)
- exponent
- remainder
- length

Measures of periodicity

1. The number of runs
2. The sum of exponents of runs
3. The total run length

The number of runs

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

Number of runs $= 19$

The number of runs

Notation:

- $runs(w)$ – the number of runs in the word w
- $runs(n) \; = \; \max\{runs(w) : \; |w| = n\}$

Conjecture (Kolpakov, Kucherov 1999)

$$runs(n) \; \leq \; n$$

The number of runs

Theorem (Crochemore et al. 2008, Kusano et al. 2008)

$$0.944542\, n \;\leq\; runs(n) \;\leq\; 1.048\, n$$

Theorem (Crochemore et al. 2010)

$$0.41\, n \;\leq\; cubic\_runs(n) \;\leq\; 0.5\, n$$

Standard words

Standard word

- $\gamma \; = (\gamma_0, \gamma_1, \ldots, \gamma_n)$
- $w \; = \; \mathrm{Sw}(\gamma)$

Runs (Baturo, Piatkowski, Rytter 2008)

- Compact formula for $runs(w)$.
- Algorithm for $runs(w)$ with time complexity $O(|\gamma|)$.
- $\lim\limits_{|w| \longrightarrow \infty} runs(w) \; = \; 0.8\,|w|$.

Standard words

Standard word

- $\gamma \;= (\gamma_0, \gamma_1, \ldots, \gamma_n)$
- $w \;=\; \mathrm{Sw}(\gamma)$

Cubic runs (Piatkowski, Rytter 2011)

- Compact formula for $cubic\_runs(w)$.
- Algorithm for $cubic\_runs(w)$ with time complexity $O(|\gamma|)$.
- $\lim\limits_{|w| \longrightarrow \infty} cubic\_runs(w) \;\approx\; 0.36924841\,|w|$.

The sum of exponents

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

Sum of exponents $= 49\frac{23}{70} \approx 49.3286$

The sum of exponents

Notation:

- $se(w)$ – the sum of exponents in the word w
- $se(n) \ = \ \max\{sum\_exp(w) : \ |w| = n\}$

Conjecture (Kolpakov, Kucherov 1999)

$$se(n) \ \leq \ 2n$$

Theorem (Crochemore et al. 2011)

$$2.035\, n \ \leq \ se(n) \ \leq \ 4.1\, n$$

The total run length

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

Total run length $= 138$

The total run length

Notation:

- $trl(w)$ – the total run length of the word w
- $trl(n) \;=\; \max\{trl(w) : \; |w| = n\}$

Theorem (Glen, Simpson 2013)

$$\frac{n^2}{8} \;<\; trl(n) \;<\; \frac{47n^2}{72} + 2n$$

Morphisms

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \qquad h_i : \begin{cases} a \longrightarrow a^{\gamma_i} b \\ b \longrightarrow a \end{cases}, \quad \text{for } 0 \leq i \leq n$$

Morphisms

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \qquad h_i : \begin{cases} a \longrightarrow a^{\gamma_i} b \\ b \longrightarrow a \end{cases}, \quad \text{for } 0 \le i \le n$$

Lemma

1. $\mathrm{Sw}(\gamma_n) \;=\; h_n(a)$
2. $\mathrm{Sw}(\gamma_i, \gamma_{i+1}, \ldots, \gamma_n) \;=\; h_i\Big(\mathrm{Sw}(\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_n)\Big)$

Morphisms

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \qquad h_i : \begin{cases} a \longrightarrow a^{\gamma_i} b \\ b \longrightarrow a \end{cases}, \quad \text{for } 0 \le i \le n$$

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Observation

1. $\mathrm{Sw}(\gamma_0, \gamma_1, \ldots, \gamma_n) = h_0 \circ h_1 \circ \ldots \circ h_n(a)$
2. $\mathrm{Sw}(\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_n) = h_i^{-1}\Big(\mathrm{Sw}(\gamma_i, \gamma_{i+1}, \ldots, \gamma_n)\Big)$

Example

$$\gamma = (1,2,1,3,1) \qquad h_i : \begin{cases} a \longrightarrow a^{\gamma_i} b \\ b \longrightarrow a \end{cases} \quad \text{for } 0 \leq i \leq n$$

Example

$$\gamma = (1, 2, 1, 3, 1) \qquad h_4 : \begin{cases} a \longrightarrow ab \\ b \longrightarrow a \end{cases}$$

$$\mathrm{Sw}(1) = h_4(a) \qquad = ab$$

Example

$$\gamma = (1, 2, 1, 3, 1) \qquad h_3 : \begin{cases} a \longrightarrow aaab \\ b \longrightarrow a \end{cases}$$

$$\mathrm{Sw}(1) = h_4(a) = ab$$

$$\mathrm{Sw}(3, 1) = h_3\big(\mathrm{Sw}(1)\big) = aaaba$$

Example

$$\gamma = (1, 2, 1, 3, 1) \qquad h_2 : \begin{cases} a \longrightarrow ab \\ b \longrightarrow a \end{cases}$$

$$\begin{aligned} \mathrm{Sw}(1) = h_4(a) &= ab \\ \mathrm{Sw}(3,1) = h_3\big(\mathrm{Sw}(1)\big) &= aaaba \\ \mathrm{Sw}(1,3,1) = h_2\big(\mathrm{Sw}(3,1)\big) &= abababaab \end{aligned}$$

Example

$$\gamma = (1, 2, 1, 3, 1) \qquad h_1 : \begin{cases} a \longrightarrow aab \\ b \longrightarrow a \end{cases}$$

$$\begin{aligned}
\mathrm{Sw}(1) = h_4(a) &= ab \\
\mathrm{Sw}(3,1) = h_3(\mathrm{Sw}(1)) &= aaaba \\
\mathrm{Sw}(1,3,1) = h_2(\mathrm{Sw}(3,1)) &= abababaab \\
\mathrm{Sw}(2,1,3,1) = h_1(\mathrm{Sw}(1,3,1)) &= aabaaabaaabaaabaaba
\end{aligned}$$

Example

$$\gamma = (1, 2, 1, 3, 1) \qquad h_0 : \begin{cases} a \longrightarrow ab \\ b \longrightarrow a \end{cases}$$

$$\begin{aligned}
\mathrm{Sw}(1) = h_4(a) &= ab \\
\mathrm{Sw}(3,1) = h_3\big(\mathrm{Sw}(1)\big) &= aaaba \\
\mathrm{Sw}(1,3,1) = h_2\big(\mathrm{Sw}(3,1)\big) &= abababaab \\
\mathrm{Sw}(2,1,3,1) = h_1\big(\mathrm{Sw}(1,3,1)\big) &= aabaaabaaabaaabaaba \\
\mathrm{Sw}(1,2,1,3,1) = h_0\big(\mathrm{Sw}(2,1,3,1)\big) &= ababaababaababaababaabab
\end{aligned}$$

Example

$$\gamma = (1,2,1,3,1) \qquad h_0 : \begin{cases} a \longrightarrow ab \\ b \longrightarrow a \end{cases}$$

$$\begin{aligned}
\mathrm{Sw}(1) = h_4(a) &= ab \\
\mathrm{Sw}(3,1) = h_3\big(\mathrm{Sw}(1)\big) &= aaaba \\
\mathrm{Sw}(1,3,1) = h_2\big(\mathrm{Sw}(3,1)\big) &= abababaab \\
\mathrm{Sw}(2,1,3,1) = h_1\big(\mathrm{Sw}(1,3,1)\big) &= aabaaabaaabaaabaaba \\
\mathrm{Sw}(1,2,1,3,1) = h_0\big(\mathrm{Sw}(2,1,3,1)\big) &= ababaabababaababababaabababaababaab
\end{aligned}$$

Notation: $N_\gamma = |\mathrm{Sw}(\gamma_i, \ldots, \gamma_n)|_a$

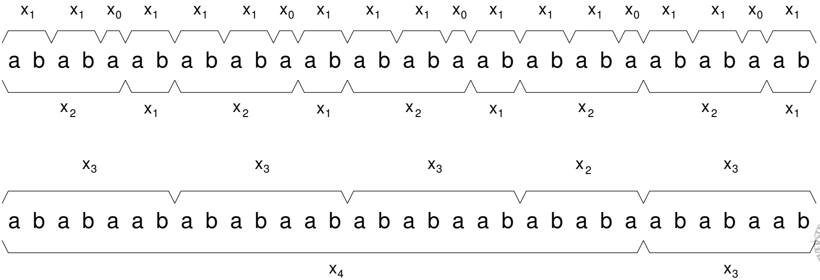
m-partition

Every standard word $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ can be represented in one of the forms:

$$(i) \quad x_m^{\alpha_1}\, x_{m-1}\, x_m^{\alpha_2}\, x_{m-1} \cdots x_m^{\alpha_s}\, x_{m-1}\, x_m$$

$$(ii) \quad x_m^{\beta_1}\, x_{m-1}\, x_m^{\beta_2}\, x_{m-1} \cdots x_m^{\beta_s}\, x_{m-1}$$

where $\alpha_k,\ \beta_k\ \in\ \{\gamma_m,\ \gamma_m + 1\}$, $0 \le m \le n$.



Proposition

The structure of occurrences of the block x_m (respectively x_{m-1}) in the m-partition of $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ corresponds to the structure of occurrences of the letter a (respectively letter b) in $\mathrm{Sw}(\gamma_m, \ldots, \gamma_n)$.

m	m-partition	$\mathrm{Sw}(\gamma_m, \ldots, \gamma_n)$
1	$ab{\cdot}ab{\cdot}a{\cdot}ab{\cdot}ab{\cdot}ab{\cdot}a{\cdot}ab{\cdot}ab{\cdot}ab{\cdot}a{\cdot}ab{\cdot}ab{\cdot}ab{\cdot}a{\cdot}ab{\cdot}ab{\cdot}a{\cdot}ab$	$aabaaabaaabaaabaaba$
2	$ababa{\cdot}ab{\cdot}ababa{\cdot}ab{\cdot}ababa{\cdot}ab{\cdot}ababa{\cdot}ababa{\cdot}ab$	$abababaab$
3	$ababaab{\cdot}ababaab{\cdot}ababaab{\cdot}ababa{\cdot}ababaab$	$aaaba$
4	$ababaabababaabababaababababa{\cdot}ababaab$	ab

Lemma

1. The longest prefix of $x_{i-1} \cdot x_i$ with the period of the length $|x_i|$ is of the form $x_i \cdot \widehat{x_{i-1}}$.

Notation: $\widehat{x}$ – x with last two letters removed.

Lemma

1. The longest prefix of $x_{i-1} \cdot x_i$ with the period of the length $|x_i|$ is of the form $x_i \cdot \widehat{x_{i-1}}$.
2. For each occurrence of $y \cdot x_i$ in w, y is the last letter of the block x_{i-1} or x_i of the i-partition of w.

Notation: $\widehat{x}$ – x with last two letters removed.

Lemma

1. The longest prefix of $x_{i-1} \cdot x_i$ with the period of the length $|x_i|$ is of the form $x_i \cdot \widehat{x_{i-1}}$.
2. For each occurrence of $y \cdot x_i$ in w, y is the last letter of the block x_{i-1} or x_i of the i-partition of w.
3. For every $0 \leq k \leq n$ and every $1 \leq i \leq \gamma_k$ the word $(x_k)^i x_{k-1}$ is primitive.

Notation: $\widehat{x}$ – x with last two letters removed.

Lemma

1. The longest prefix of $x_{i-1} \cdot x_i$ with the period of the length $|x_i|$ is of the form $x_i \cdot \widehat{x_{i-1}}$.
2. For each occurrence of $y \cdot x_i$ in w, y is the last letter of the block x_{i-1} or x_i of the i-partition of w.
3. For every $0 \leq k \leq n$ and every $1 \leq i \leq \gamma_k$ the word $(x_k)^i x_{k-1}$ is primitive.
4. The period of each maximal repetition in a standard word $\mathrm{Sw}(\gamma_0, \gamma_1, \ldots, \gamma_n)$ is of the form x_i or $(x_i)^j x_{i-1}$, where $0 \leq i \leq n$, $0 < j < \gamma_i$.

Notation: $\widehat{x}$ – x with last two letters removed.

Computation of the sum of exponents (the total run length)

- $\mathrm{Sw}(\gamma)$ – standard word

- $se'_i(\gamma)$ – sum of exponents of runs with periods x_i
- $se''_i(\gamma)$ – sum of exponents of runs with periods $(x_i)^j x_{i-1}$

- $trl'_i(\gamma)$ – total length of runs with periods x_i
- $trl''_i(\gamma)$ – total length of runs with periods $(x_i)^j x_{i-1}$

- $se(\gamma) \;=\; \sum_{i=0}^{n} \; se'_i(\gamma) + se''_i(\gamma)$
- $trl(\gamma) \;=\; \sum_{i=0}^{n} \; trl'_i(\gamma) + trl''_i(\gamma)$

(1) $\underbrace{aaba}_{x_2}\underbrace{aaba}_{x_2}\underbrace{aaba}_{x_2}\underbrace{aab}_{x_1}\underbrace{aaba}_{x_2}\underbrace{aaba}_{x_2}\underbrace{aaba}_{x_2}\underbrace{aaba}_{x_2}\underbrace{aab}_{x_1}$

(2) $\underbrace{aaba}_{x_2}\underbrace{aaba}_{x_2}\underbrace{aaba}_{x_2}\underbrace{aab}_{x_1}\underbrace{aaba}_{x_2}\underbrace{aaba}_{x_2}\underbrace{aaba}_{x_2}\underbrace{aab}_{x_1}\underbrace{aaba}_{x_2}$

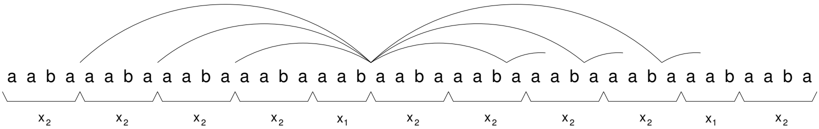
Lemma – sum of exponents

$$se'_i(\gamma) = N_\gamma(i+1)\cdot\left(\gamma_i+1+\frac{|x_{i-1}|-2}{|x_i|}\right)+\Big(N_\gamma(i+2)-1\Big)+\Delta_n(i)\frac{2}{|x_i|}$$

Notation: $\Delta_n(i) \; = \; |n-i+1| \; mod \; 2$

$$N_\gamma \; = \; |\mathrm{Sw}(\gamma_i,\ldots,\gamma_n)|_a$$

Runs with period $(x_i)^k x_{i-1}$

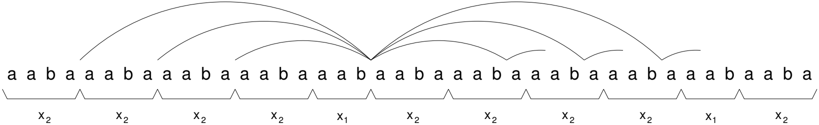


Lemma – sum of exponents

$$se''_i(\gamma) \; = \; \Big(N_\gamma(i+1)-1\Big)\cdot\sum_{k=1}^{\gamma_i-1}\left(2+\frac{|x_i|-2}{k\cdot|x_i|+|x_{i-1}|}\right)$$

Notation: $N_\gamma \; = \; |\mathrm{Sw}(\gamma_i,\ldots,\gamma_n)|_a$

Runs with period $(x_i)^k x_{i-1}$



Lemma – sum of exponents

$$se''_i(\gamma) \;=\; \Big(N_\gamma(i+1)-1\Big)\cdot\sum_{k=1}^{\gamma_i-1}\left(2+\frac{|x_i|-2}{k\cdot|x_i|+|x_{i-1}|}\right)$$

Lemma – total run length

$$trl''_i(\gamma) \;=\; \Big(N_\gamma(i+1)-1\Big)\cdot(\gamma_i-1)\frac{(\gamma_i+2)|x_i|+2|x_i|-4}{2}$$

Notation: $N_\gamma \;=\; |\mathrm{Sw}(\gamma_i,\ldots,\gamma_n)|_a$

Algorithm

$se \longleftarrow 0 \quad (trl \longleftarrow 0)$

$|x_{-1}| \longleftarrow 1, \;\; |x_0| \longleftarrow 0$

$N_\gamma(n+1) \longleftarrow 1,$

$N_\gamma(n+2) \longleftarrow 0$

for $k := 0$ **to** n **do**

$\qquad |x_k| \;\longleftarrow\; \gamma_k \, |x_{k-1}| + |x_{k-2}|$

for $k := n$ **downto** 0 **do**

$\qquad$ update se (respectively trl) values depending on γ_k

$\qquad$ update value of $|x_{k-1}|$

$\qquad$ update value of $N_\gamma(k)$

return se (respectively trl)

Time complexity

- Sum of exponents – $O(\|\gamma\|)$, where $\|\gamma\| = \gamma_0 + \gamma_1 + \cdots + \gamma_n$
- Total run length – $O(|\gamma|)$

Thank You
For Your Attention