Computing the Number of Cubic Runs in Standard Sturmian Words

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Run (maximal repetition)

$$\underbrace{\text{ababaab}}\,\underbrace{\text{ababaab}}\,\underbrace{\text{ababaab}}\,\underbrace{\text{ababaab}}\,\underbrace{\text{aba}}\,\text{ab}$$

Notation:

- $\rho(w)$ – the number of runs in the word w
- $\rho^{(3)}(w)$ – the number of cubic runs in the word w

- $\rho(n)$ – maximal number of runs in words of length n
- $\rho^{(3)}(n)$ – maximal number of cubic runs in words of length n

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

a b a b a a b a b a b a a b a b a b a a b a b a b a a b a b a a b

Sturmian words are infinite words that has exactly $n+1$ factors of length n for each $n \geq 0$.

Example:

$a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,a\,b \dots$

1. $a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,a\,b \dots$
2. $a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,a\,b \dots$
3. $a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,a\,b \dots$
4. $a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,b\,a\,a\,b\,a\,b\,a\,a\,b \dots$

Standard word

- Alphabet: $\Sigma = \{a, b\}$
- Directive sequence: $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$, where $\gamma_0 \geq 0$ and $\gamma_1, \ldots, \gamma_n > 0$
- Recurrence:
  - $x_{-1} = b$
  - $x_0 = a$
  - $x_k = (x_{k-1})^{\gamma_{k-1}} \cdot x_{k-2}$
- $\mathrm{Sw}(\gamma_0, \gamma_1, \ldots, \gamma_n) = x_{n+1}$

Standard word example

- $\gamma \;=\; (1,2,1,3,1)$

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- $x_1 = (x_0)^1 \cdot x_{-1} = a \cdot b$

Standard word example

- $\gamma \;=\; (1, 2, 1, 3, 1)$

- $x_{-1} \;=\; b$
- $x_0 = a$
- $x_1 \;=\; (x_0)^1 \cdot x_{-1} \;=\; a \cdot b$
- $x_2 \;=\; (x_1)^2 \cdot x_0 \;=\; ab \cdot ab \cdot a$

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- $x_3 \;=\; (x_2)^1 \cdot x_1 \;=\; ababa \cdot ab$
- $x_4 \;=\; (x_3)^3 \cdot x_2 \;=\; ababaab \cdot ababaab \cdot ababaab \cdot ababa$

Standard word example

- $\gamma \;=\; (1, 2, 1, 3, {\color{red}1})$

- $x_{-1} \;=\; b$
- $x_0 = a$
- $x_1 \;=\; (x_0)^1 \cdot x_{-1} \;=\; a \cdot b$
- $x_2 \;=\; (x_1)^2 \cdot x_0 \;=\; ab \cdot ab \cdot a$
- $x_3 \;=\; (x_2)^1 \cdot x_1 \;=\; ababa \cdot ab$
- $x_4 \;=\; (x_3)^3 \cdot x_2 \;=\; ababaab \cdot ababaab \cdot ababaab \cdot ababa$
- $x_5 \;=\; (x_4)^{\color{red}1} \cdot x_3 \;=\; ababaababababaababababaababababa \cdot ababaab$

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- $\mathrm{Sw}(1,2,1,3,1) \;=\; ababaababaababaababaababaababaab$

Morphisms

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \qquad h_i : \begin{cases} a \longrightarrow a^{\gamma_i} b \\ b \longrightarrow a \end{cases}, \quad \text{for } 0 \le i \le n$$

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Lemma

1. $\mathrm{Sw}(\gamma_n) \;=\; h_n(a)$
2. $\mathrm{Sw}(\gamma_i, \gamma_{i+1}, \ldots, \gamma_n) \;=\; h_i\Big(\mathrm{Sw}(\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_n)\Big)$

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Observation

1. $\mathrm{Sw}(\gamma_0, \gamma_1, \ldots, \gamma_n) \ = \ h_0 \circ h_1 \circ \ldots \circ h_n(a)$
2. $\mathrm{Sw}(\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_n) \ = \ h_i^{-1}\Big(\mathrm{Sw}(\gamma_i, \gamma_{i+1}, \ldots, \gamma_n)\Big)$

Example

$$\gamma = (1, 2, 1, 3, 1) \qquad h_i : \begin{cases} a \longrightarrow a^{\gamma_i} b \\ b \longrightarrow a \end{cases} \quad \text{for } 0 \leq i \leq n$$

Example

$$\gamma \;=\; (1,2,1,3,1) \qquad h_4 : \begin{cases} a \longrightarrow ab \\ b \longrightarrow a \end{cases}$$

$$\mathrm{Sw}(1) = h_4(a) \qquad = ab$$

Example

$$\gamma = (1, 2, 1, 3, 1) \qquad h_3 : \begin{cases} a \longrightarrow aaab \\ b \longrightarrow a \end{cases}$$

$$\mathrm{Sw}(1) = h_4(a) = ab$$

$$\mathrm{Sw}(3,1) = h_3\big(\mathrm{Sw}(1)\big) = aaaba$$

Example

$$\gamma = (1, 2, 1, 3, 1) \qquad h_2 : \begin{cases} a \longrightarrow ab \\ b \longrightarrow a \end{cases}$$

$$\mathrm{Sw}(1) = h_4(a) = ab$$

$$\mathrm{Sw}(3,1) = h_3\big(\mathrm{Sw}(1)\big) = aaaba$$

$$\mathrm{Sw}(1,3,1) = h_2\big(\mathrm{Sw}(3,1)\big) = abababaab$$

Example

$$\gamma = (1, 2, 1, 3, 1) \qquad h_1 : \begin{cases} a \longrightarrow aab \\ b \longrightarrow a \end{cases}$$

$$\mathrm{Sw}(1) = h_4(a) = ab$$

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$$\mathrm{Sw}(1,3,1) = h_2\big(\mathrm{Sw}(3,1)\big) = abababaab$$

$$\mathrm{Sw}(2,1,3,1) = h_1\big(\mathrm{Sw}(1,3,1)\big) = aabaaabaaabaaabaaba$$

Example

$$\gamma = ({\color{red}1}, 2, 1, 3, 1) \qquad h_0 : \begin{cases} a \longrightarrow ab \\ b \longrightarrow a \end{cases}$$

$$\begin{aligned}
\mathrm{Sw}(1) = h_4(a) &= ab \\
\mathrm{Sw}(3,1) = h_3\big(\mathrm{Sw}(1)\big) &= aaaba \\
\mathrm{Sw}(1,3,1) = h_2\big(\mathrm{Sw}(3,1)\big) &= abababaab \\
\mathrm{Sw}(2,1,3,1) = h_1\big(\mathrm{Sw}(1,3,1)\big) &= aabaaabaaabaaabaaba \\
\mathrm{Sw}(1,2,1,3,1) = h_0\big(\mathrm{Sw}(2,1,3,1)\big) &= ababaabababaabababaabababaababaab
\end{aligned}$$

Lemma

The period of each cubic run in the standard word $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ *is of the form* x_i *for* $0 \le i \le n$.

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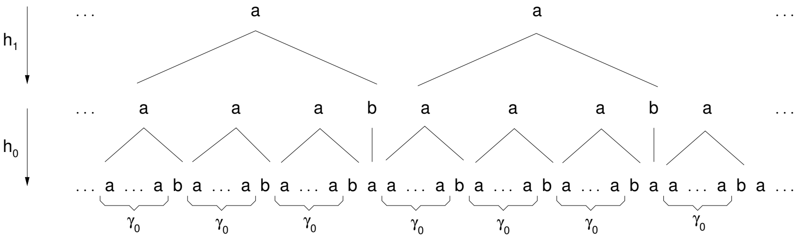
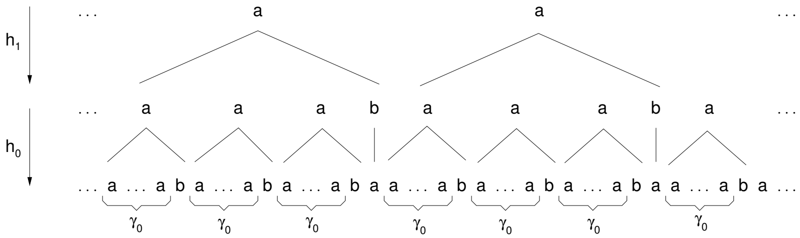
Types of cubic runs

- **short** – periods of the form x_0 and x_1 (a and $a^k b$)
- **medium** – periods of the form x_2
- **long** – periods of the form x_i for $3 \leq i \leq n$

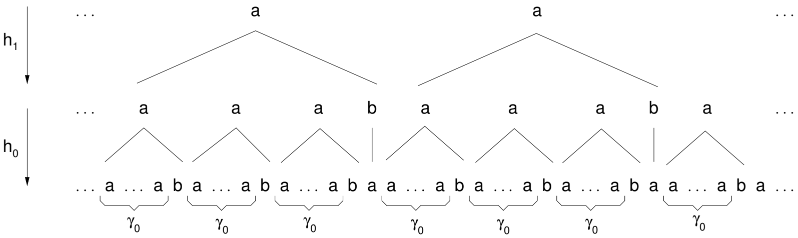


Lemma (short cubic runs with period a)



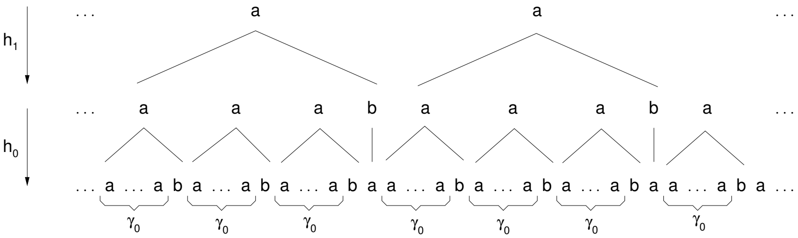
Lemma (short cubic runs with period a)

$$\rho^{(3)}_{S_1}\Big(\mathrm{Sw}(\gamma_0,\ldots,\gamma_n)\Big) = \begin{cases} \\ \Big|\mathrm{Sw}(\gamma_1,\ldots,\gamma_n)\Big|_a & \text{for} \quad \gamma_0 > 2 \end{cases}$$



Lemma (short cubic runs with period a)

$$\rho^{(3)}_{S_1}\Big(\mathrm{Sw}(\gamma_0,\ldots,\gamma_n)\Big) = \begin{cases} \Big|\mathrm{Sw}(\gamma_2,\ldots,\gamma_n)\Big|_a - odd(n) & \text{for} \quad \gamma_0 = 2 \\ \Big|\mathrm{Sw}(\gamma_1,\ldots,\gamma_n)\Big|_a & \text{for} \quad \gamma_0 > 2 \end{cases}$$



Lemma (short cubic runs with period a)

$$\rho^{(3)}_{S_1}\Big(\mathrm{Sw}(\gamma_0,\dots,\gamma_n)\Big) = \begin{cases} 0 & \text{for} \quad \gamma_0 = 1 \\ \Big|\mathrm{Sw}(\gamma_2,\dots,\gamma_n)\Big|_a - odd(n) & \text{for} \quad \gamma_0 = 2 \\ \Big|\mathrm{Sw}(\gamma_1,\dots,\gamma_n)\Big|_a & \text{for} \quad \gamma_0 > 2 \end{cases}$$

Lemma (short cubic runs with period $a^k b$)

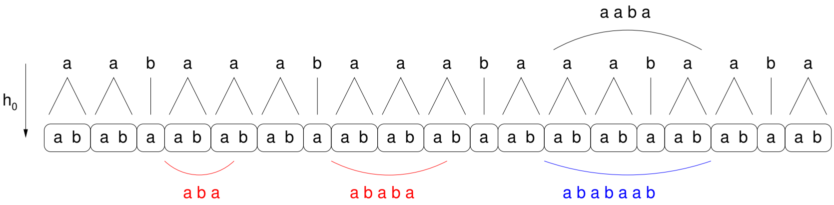
$$\rho^{(3)}_{S_2}\Big(\mathrm{Sw}(\gamma_0,\ldots,\gamma_n)\Big) = \begin{cases} 0 & \text{for} \quad \gamma_1 = 1 \\ \Big|\mathrm{Sw}(\gamma_3,\ldots,\gamma_n)\Big|_a - even(n) & \text{for} \quad \gamma_1 = 2 \\ \Big|\mathrm{Sw}(\gamma_2,\ldots,\gamma_n)\Big|_a & \text{for} \quad \gamma_1 > 2 \end{cases}$$

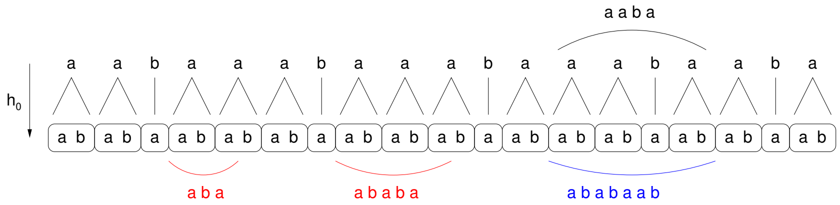
Lemma (medium cubic runs for $n \geq 3$)

$$\rho_M^{(3)}\Big(\mathrm{Sw}(\gamma_0,\ldots,\gamma_n)\Big) = \begin{cases} \Big|\mathrm{Sw}(\gamma_4,\ldots,\gamma_n)\Big|_a - 1 & \text{for} \quad \gamma_2 = 1 \\ \Big|\mathrm{Sw}(\gamma_3,\ldots,\gamma_n)\Big|_a & \text{for} \quad \gamma_2 \geq 2 \end{cases}$$

Lemma (medium cubic runs for $n = 2$)

$$\rho_M^{(3)}\Big(\mathrm{Sw}(\gamma_0,\gamma_1,\gamma_2)\Big) = \begin{cases} 0 & \text{for} \quad \gamma_2 \leq 2 \\ 1 & \text{for} \quad \gamma_2 > 2 \end{cases}$$


a a b a
a a b a a a b a a a b a a a b a a b a
h0
ab ab a ab ab ab a ab ab ab a ab ab ab a ab ab a ab
a b a
a b a b a
a b a b a a b



Definition

A factor x of the word w is synchronized with the morphpism h iff each occurrence of x in w starts at the beginning of some h-block and ends at the end of some h-block.

Lemma

The periods of large cubic runs in the standard word $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ are synchronized with the morhpism h_0.

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Lemma

$$\rho_L^{(3)}(w) \;=\; \rho_M^{(3)}\Big(h^{-1}(w)\Big) \;+\; \rho_L^{(3)}\Big(h^{-1}(w)\Big)$$

Theorem

Let $w = \mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ *and* $w_i = \mathrm{Sw}(\gamma_i, \ldots, \gamma_n)$ *be standard words. Then*

$$\rho^{(3)}(w) \ = \ \rho^{(3)}_{S_1}(w) \ + \ \rho^{(3)}_{S_1}(w) \ + \ \sum_{i=0}^{n-2} \rho^{(3)}_{M}(w_i).$$

Theorem

We can count the number $\rho^{(3)}\Big(\mathrm{Sw}(\gamma_0,\ldots,\gamma_n)\Big)$ *in time linear with respect to* $|(\gamma_0,\ldots,\gamma_n)|$, *hence logarithmic time with respect to* $\Big|\mathrm{Sw}(\gamma_0,\ldots,\gamma_n)\Big|$.

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Observation

$$\Big|\mathrm{Sw}(\gamma_i,\ldots,\gamma_n)\Big|_a \;=\; \gamma_i\cdot\Big|\mathrm{Sw}(\gamma_{i+1},\ldots,\gamma_n)\Big|_a+\Big|\mathrm{Sw}(\gamma_{i+2},\ldots,\gamma_n)\Big|_a$$

Crochemore, Iliopoulos, Kubica, Radoszewski, Rytter, Waleń 2010

$$0.41\, n \;\leq\; \rho^{(3)}(n) \;\leq\; 0.5\, n$$

Crochemore, Iliopoulos, Kubica, Radoszewski, Rytter, Waleń 2010

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$$\frac{\rho^{(3)}(F_n)}{|F_n|} \longrightarrow 0.763932\ldots$$

Theorem

For any standard word w we have:

$$\rho^{(3)}(w) \;\leq\; 0.36924841\,|w|,$$

and there is strictly growing sequence $\{w_k\}$ of standard words such that

$$\lim_{k\to\infty} \frac{\rho^{(3)}(w_k)}{|w_k|} \;=\; \frac{5\Phi+3}{13\Phi+9} \;\approx\; 0.36924841,$$

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Idea of the proof

$$w_k \ = \ (3, 2, \underbrace{1, 1, \ldots, 1}_{k})$$

Thank You
For Your Attention