# Translating Between Wavelet Tree and Wavelet Matrix Construction

Patrick Dinklage

TU Dortmund University
Chair of Algorithm Engineering (LS11)
Otto-Hahn-Straße 14
44227 Dortmund
Germany
patrick.dinklage@tu-dortmund.de

**Abstract.** The wavelet tree (Grossi et al. [SODA, 2003]) and wavelet matrix (Claude et al. [Inf. Syst., 2015]) are compact data structures with many applications such as text indexing or computational geometry. By continuing the recent research of Fischer et al. [ALENEX, 2018], we explore the similarities and differences of these heavily related data structures with focus on their construction. We develop a data structure to modify construction algorithms for either the wavelet tree or matrix to construct instead the other. This modification is *efficient*, in that it does not worsen the asymptotic time and space requirements of any known wavelet tree or wavelet matrix construction algorithm.

**Keywords:** text indexing, wavelet tree, wavelet matrix

## 1 Introduction

The wavelet tree [5] is a data structure with numerous applications in text indexing, data compression, computational geometry (as an alternative to fractional cascading) and other areas [3, 8, 10]. Common queries that the wavelet tree can answer efficiently are *rank* and *select* for any symbol that occurs in the underlying text, as well as *access* queries to restore said text. The wavelet matrix [2] is a related data structure with the same asymptotic running times for these queries However, they are faster in practice, because they require less subqueries on bit vectors to be answered.

Both data structures are based on storing $n\lceil\log\sigma\rceil$ bits for the text of length $n$ over an alphabet of size $\sigma$ and answer access, rank and select queries in asymptotic time $\mathcal{O}(\log\sigma)$. Since they can also be used for accessing individual characters in time $\mathcal{O}(\log\sigma)$, they can both be seen as different encodings of the text. They differ (a) in the order these bits are stored, and (b) in the auxiliary data required to answer the queries. However, there are many similarities between these two data structures and it is natural to ask how far these similarities go. In this work, we focus on the construction process of the data structures.

**Related work.** Fischer et al. [4, Sect. 5.2] recently showed that there is a data structure to efficiently transform any construction algorithm for the wavelet tree to construct instead the wavelet matrix without worsening the asymptotic construction times. This makes it possible to apply techniques used by (parallel) wavelet tree construction algorithms, which make use of the tree structure, to the wavelet matrix, which discards the tree structure. Their data structure occupies $\mathcal{O}(n+\sigma\log n)$ bits of space and can be constructed in time $\mathcal{O}(n+\sigma)$ using $o(n+\sigma)$ bits of memory.

Patrick Dinklage: Translating Between Wavelet Tree and Wavelet Matrix Construction, pp. 126–135.
Proceedings of PSC 2019, Jan Holub and Jan Žďárek (Eds.), ISBN 978-80-01-06618-8 © Czech Technical University in Prague, Czech Republic

**Our contributions.** Fischer et al. left open whether there is a data structure for the inverse direction, i.e., whether there is an efficient way to construct the wavelet tree using a construction algorithm for the wavelet matrix. In order to learn more about the similarities and differences between the two, we propose a first solution to this problem and give the corresponding data structure of the same asymptotic space requirements as that in [4]. It can be constructed easily in time $\mathcal{O}(\sigma)$ from the text's histogram and its principle works for both directions. However, there is a slight limitation that gives us some insight on the different information contained in the wavelet tree and matrix.

## 2 Preliminaries

Let $T \in \Sigma^n$ be a text over an alphabet $\Sigma$. For some integer $i < n$, let $T[i]$ be the $i$-th symbol of $T$. We use zero-based indexing, so that $T[0]$ is the first symbol of $T$ and $T[n-1]$ is the last.

**Computational model.** We use the *word RAM* model, where we assume that we can perform arithmetic operations on words of bit width $\mathcal{O}(\log n)$ in constant time.

**Histogram.** The *histogram* $H : c \mapsto \mathrm{occ}_T(c)$ of $T$ maps each symbol $c \in \Sigma$ to its number $\mathrm{occ}_T(c)$ of occurrences in $T$. The set of those $\sigma$ symbols with $\mathrm{occ}_T(c) > 0$ are the *effective alphabet* of $T$. We represent it as the interval $\Sigma' = [0, \sigma)$, so that the lexicographically smallest symbol is represented by 0 and the largest symbol by $\sigma - 1$. Let $\mathrm{eff}_T(c) \in \Sigma'$ be the rank of $c$ in the effective alphabet. In the *effective transformation* $T'$ of $T$, we set $T'[i] \coloneqq \mathrm{eff}_T(T[i])$ for each $i < n$. As an example, consider the text and alphabet in Figure 1. The effective transformation of the text is $T' = 6\,0\,5\,1\,2\,1\,4\,4\,3\,1\,1$.

**C array.** For every $x \in \Sigma'$, the $C$ *array* contains the accumulated number of occurrences of symbols in $T'$ that are lexicographically smaller than $x$. Formally, it is $C[x] \coloneqq \sum_{k=0}^{x-1} \mathrm{occ}_{T'}(k)$. We furthermore define $C[\sigma] \coloneqq n$.

**Bit vectors.** A *bit vector* is a text over the binary alphabet $\mathbb{B} = \{0, 1\}$. Let $B = \mathbb{B}^n$ be a bit vector of length $n$. For every position $i < n$, the function $\mathrm{rank}_1(B, i)$ returns the number of 1-bits in $B$ from its beginning up to (including) position $i$. For a $k > 0$, the function $\mathrm{select}_1(B, k)$ returns the position of the $k$-th 1-bit in $B$. The functions $\mathrm{rank}_0$ and $\mathrm{select}_0$ are defined analogously for 0-bits. There is a data structure that can answer rank and select queries for a fixed $B$ and any $i$ or $k$, respectively, in time $\mathcal{O}(1)$, requires $o(n)$ bits of memory and can be constructed in time $\mathcal{O}(n)$ [6].

**Bit reversal.** Let $B \in \mathbb{B}^*$ be a bit vector and let $(B)_{\mathbb{N}} \in \mathbb{N}$ denote the integer that $B$ is the binary representation of. For $k > 0$ and an integer $i < 2^k$, we call $(i)_{\mathbb{B},k} \in \mathbb{B}^k$ the $k$-bit binary representation of $i$. Let $B^R$ denote the reversal of $B$. We define the *$k$-bit reversal* $\mathrm{bitrev}_k(i) \coloneqq (((i)_{\mathbb{B},k})^R)_{\mathbb{N}}$ as the integer represented by the reversal of $i$'s $k$-bit binary representation. For a fixed $k$, the *bit-reversal permutation* maps each integer $i < 2^k$ to its $k$-bit reversal. To give examples, Table 1 shows the bit-reversal permutations for $k = 2$ and $k = 3$.

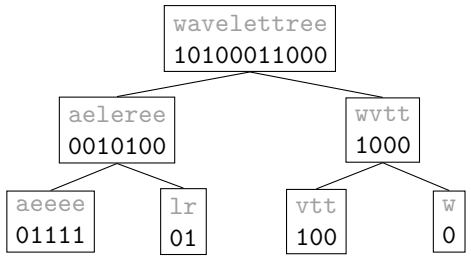
| $i$ | $(i)_{\mathbb{B},2}$ | $((i)_{\mathbb{B},2})^R$ | $\text{bitrev}_2(i)$ |
|---|---|---|---|
| 0 | 00 | 00 | 0 |
| 1 | 01 | 10 | 2 |
| 2 | 10 | 01 | 1 |
| 3 | 11 | 11 | 3 |

(a) Bit-reversal permutation for $k = 2$.

| $i$ | $(i)_{\mathbb{B},3}$ | $((i)_{\mathbb{B},3})^R$ | $\text{bitrev}_3(i)$ |
|---|---|---|---|
| 0 | 000 | 000 | 0 |
| 1 | 001 | 100 | 4 |
| 2 | 010 | 010 | 2 |
| 3 | 011 | 110 | 6 |
| 4 | 100 | 001 | 1 |
| 5 | 101 | 101 | 5 |
| 6 | 110 | 011 | 3 |
| 7 | 111 | 111 | 7 |

(b) Bit-reversal permutation for $k = 3$.

Table 1: Breakdowns of the bit-reversal permutations for $k = 2$ (left) and $k = 3$ (right). The first column contains the integers $i < 2^k$, the second shows their $k$-bit binary representations, the third shows the reversals and the final column contains the $k$-bit reversal of $i$.




$\Sigma = \{\texttt{a}, \texttt{e}, \texttt{l}, \texttt{r}, \texttt{t}, \texttt{v}, \texttt{w}\}$
$\Sigma' = [0, 7)$

| $c$ | $\text{eff}_T(c)$ |
|---|---|
| a | $0 = 000_b$ |
| e | $1 = 001_b$ |
| l | $2 = 010_b$ |
| r | $3 = 011_b$ |
| t | $4 = 100_b$ |
| v | $5 = 101_b$ |
| w | $6 = 110_b$ |

Figure 1: The wavelet tree (left), alphabet, effective alphabet and binary representations of symbols (right) for $T = \texttt{wavelettree}$. The texts above the node bit vectors are shown only for comprehensibility; they are not a part of the node labels and are not stored.

### 2.1 The Wavelet Tree

The *wavelet tree* [5] is a binary tree of height $\lceil \log \sigma \rceil$ where each node $v$ represents an interval $[a, b] \subseteq \Sigma'$ of the effective alphabet and is labeled by a bit vector $B_v \in \mathbb{B}^+$. $B_v$ contains one bit for each text position $i$, in text order, where $T'[i] \in [a, b]$: a 0-bit if $T'[i] \leq \lfloor \frac{a+b}{2} \rfloor$, i.e., if the symbol $T'[i]$ lies in the left half of the represented interval, or a 1-bit otherwise.

The root node represents the entire effective alphabet $\Sigma'$ and thus its bit vector has length $n$. A node $v$ has two children iff $a < b$. We apply the described structure recursively for the left child to represent the interval $[a, \lfloor \frac{a+b}{2} \rfloor]$ (the *left half*) and the right child to represent $[\lfloor \frac{a+b}{2} \rfloor + 1, b]$ (the *right half*). Following that, the tree's leaves are those nodes that represent an interval of size one, i.e., precisely one symbol from the input alphabet ($a = b$). Since the bit vector of a leaf contains only zero-bits, we need not store level $\lceil \log \sigma \rceil + 1$ of the wavelet tree, because it would consist of leaves only. Figure 1 shows an example of a wavelet tree.

The size of any node in the wavelet tree, i.e., the length of its bit vector label, can be precomputed using the $C$ array:

**Observation 1** *Let $[a, b] \subseteq \Sigma'$ be the alphabet interval represented by a wavelet tree node $v$. The length of the bit vector $B_v$ that labels $v$ is $|B_v| = C[b + 1] - C[a]$.*

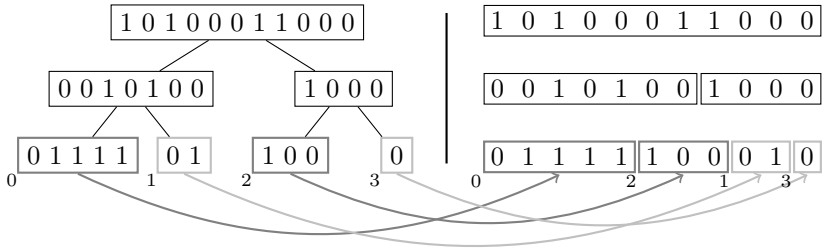



Figure 2: Comparison of the node ordering in the wavelet tree (left) and the wavelet matrix (right). Due to the nature of the bit reversal permutation, the ordering on the first two levels remains the same in the wavelet matrix. On the third level, we observe how nodes 0 and 2 (left children of their respective parents) go to the left part of the corresponding wavelet matrix bit vector and nodes 1 and 3 (right children of their respective parents) go to the right.

For storing the wavelet tree, we consider the *pointerless* representation (also known as the *levelwise* representation), where we concatenate the bit vectors on each level and enhance them by constant-time rank/select support. This is is enough information to be able to navigate in the tree [10]. The concatenation of bit vectors on any level has a length of at most $n$ bits, so that the wavelet tree's bit vectors consume at most $n\lceil \log \sigma \rceil$ bits in total.

### 2.2 The Wavelet Matrix

The *wavelet matrix* [2] can be thought of as an alternative representation of the pointerless wavelet tree. In the wavelet tree, in order to retrieve the bit vector $B_\ell^T$ for level $\ell$, we concatenate the bit vectors of the single nodes on that level from *left to right*. In the wavelet matrix, the nodes are concatenated in a different order to obtain bit vector $B_\ell^M$: all left children of their respective parents are moved to the left and all right children are moved to the right. Like in the pointerless wavelet tree, we concatenate the bit vectors of all nodes on every level. Figure 2 shows an example. The re-ordering of nodes corresponds to the bit-reversal permutation of the node ranks on the respective level [4].

A practical consequence of the different ordering is that navigation in the wavelet matrix becomes easier than in the pointerless wavelet tree. In the tree, we need to keep track of the current node's interval — its left and right boundary — within the respective level's bit vector while navigating. This can be done using two rank queries on the respective bit vector when navigating from a node to either child. In the matrix, the simpler structure makes it feasible to precompute the left boundary for the right children on each level, all of which have been concatenated in the right part of the level's bit vector. This boundary is often referred to as value $z$ in literature, as it corresponds to the number of zero bits in the bit vector. We can store $z$ for all levels using negligible $\mathcal{O}(\log \sigma \log n)$ bits and use it to save one rank query on each level while navigating.

One could precompute the same information for the wavelet tree. However, this would require us to store the left boundary of every node, resulting in $\mathcal{O}(\sigma \log n)$ bits as there are $\mathcal{O}(\sigma)$ nodes. For this reason, the wavelet matrix can be considered more relevant for practical applications where the alphabet is large.

# 3 Wavelet Tree and Wavelet Matrix Construction

We continue the research of Fischer et al. [4] and are interested in how a construction algorithm for the wavelet tree or matrix can be modified efficiently to construct the other. We consider such a modification *efficient* if the asymptotic time and space boundaries of the modified construction algorithm are not worsened. Fischer et al. show that there is a data structure that can be used to efficiently transform any construction algorithm for the wavelet tree to construct instead the wavelet matrix. We propose a data structure for the inverse direction, transforming a wavelet matrix construction algorithm to one for the wavelet tree, with the same asymptotic space requirements.

Formally, let us consider the situation where, during the construction of the wavelet tree, the $i$-th bit is set in bit vector $B_\ell^T$ of level $\ell$ of (assuming, without loss of generality, the pointerless representation). Fischer et al. [4] present a data structure to efficiently compute a function $f : (\ell, i) \mapsto (\ell, j)$ so that $j$ is the corresponding position for the bit to be set in bit vector $B_\ell^M$ of the wavelet matrix. That is, by modifying the wavelet tree constructor to set the bit at position $f(\ell, i)$ instead of $i$ on level $\ell$, it instead constructs the wavelet matrix. Because $f$ can be computed in constant time, there is no asymptotic overhead. For input length $n$ and alphabet size $\sigma$, their data structure occupies $n + \sigma + (\sigma + 2)\lceil \log n \rceil$ bits of space and can be constructed in time $\mathcal{O}(n + \sigma)$ using $o(n + \sigma)$ bits of memory, not worsening the asymptotic construction time and space requirements for any known wavelet tree constructor.

In the following, we first observe various properties of the wavelet tree that lead to a similar result for $f$ as that of [4]. Based on these observations, we develop a novel data structure for the inverse $f^{-1}$, which maps $(\ell, j)$ back to $(\ell, i)$ with the same asymptotic time and space boundaries as for $f$.

## 3.1 Locating Nodes and Bit Offsets

As previously noted, the wavelet matrix can also be represented as a tree by re-ordering the nodes of the wavelet tree on each level according to the bit-reversal permutation. Even though there are no practical advantages of storing the wavelet matrix as a tree, the notion will help us develop our data structures.

The simple nature of the re-ordering makes it easy to translate a node *ID* (the node's rank in a breadth-first traversal of the tree) between the two data structures. Based on this, we employ the following strategy to find data structures for functions $f$ and $f^{-1}$: given the level and position of the bit to be written, we attempt to find

(1) the ID of the node that the bit belongs to, and
(2) the position of the node's first bit in its level's bit vector.

Once this information is available, $f$ and $f^{-1}$ are easy to compute in constant time.

| $c$ | a | e | l | r | t | v | w | $\top$ |
|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $\mathrm{occ}_T(c)$ | 1 | 4 | 1 | 1 | 2 | 1 | 1 | 0 |
| $C[c]$ | 0 | 1 | 5 | 6 | 7 | 9 | 10 | 11 | 11 |

Figure 3: The histogram and the $C$ array for $T = \texttt{wavelettree}$. We added the artificial symbol $\top$ so $\sigma = 8$ is a power of two. The new symbol never occurs in $T$ and is lexicographically larger than the other symbols.

**Bottom level node sizes.** Observation 1 shows the relation between the $C$ array and the sizes of the wavelet tree's nodes. This relation is especially interesting regarding the *virtual* bottom-most level $h = \lceil \log \sigma \rceil$ of a full binary wavelet tree. We call this level virtual, because all bits on it would be zero and there is no need to actually store it. On this level, each node corresponds to a single symbol from the effective alphabet. Let node $v_c$ on level $h$ correspond to symbol $c \in \Sigma'$. We have $|B_{v_c}| = C[c+1] - C[c] = \mathrm{occ}_{T_{\text{eff}}}(c)$, i.e., the size of $v_c$ matches the number of occurrences of $c$.

This property is only valid if the wavelet tree is a *full* binary tree: if it was not, there would be leaves on level $h-1$ and not all nodes on level $h$ would exist. Without loss of generality, let us assume from now on that $\sigma = 2^h$ for some integral $h > 0$, i.e., that the alphabet size is a power of two. Then, the wavelet tree is a full binary tree. In case $\sigma$ is not a power of two, we introduce artificial symbols that never occur in the input and are lexicographically *larger* than all symbols of $\Sigma'$. This way, the empty nodes for these symbols are moved to the far right of the wavelet tree and can be ignored in the following.

**Locating in the wavelet tree.** We consider the situation where a wavelet tree constructor sets the $i$-th of bit vector $B_\ell^T$. Let $v(\ell, i)$ be the rank of the wavelet tree node on level $\ell$ to which the $i$-th bit belongs. We represent $v(\ell, i)$ relative to the number of the first node on level $\ell$, i.e., $v(\ell, 0) = 0$ and $v(\ell, n-1) = 2^\ell - 1$. This representation requires $\ell$ bits, because there are precisely $2^\ell - 1$ nodes on level $\ell$. Furthermore, let $p(\ell, v)$ be the position of the first bit in $B_\ell^T$ that belongs to node $v$ and let $\delta_v(\ell, i) \coloneqq i - p(\ell, v(\ell, i))$ be the distance of $i$ from that position.

We take a closer look at $v$ and $p$ on the virtual level $h$ and observe that

$$v(h, i) = \min\{x \mid C[x] > i\} - 1.$$

This is because each node on this level corresponds to precisely one symbol from the input alphabet and the $C$ array encodes, for every $c$, the number of symbols in the input that are lexicographically smaller than $c$. This corresponds to the accumulated sizes of the node's left siblings. An example of this relation can be seen comparing Figure 3 and Figure 4b (in row $\ell = 3$). The node that $i$ belongs to on level $h$ is left of the first node whose accumulated size — its entry in the $C$ array — exceeds $i$. We can immediately conclude that the first bit that belongs to node $v$ is located at position

$$p(h, v) = C[v].$$

How do $v$ and $p$ on level $h$ relate to those on the other levels $\ell < h$ that we are actually interested in? To answer this, we make use of the fact that our wavelet tree

is a full binary tree: the size of a node equals the sum of its children's sizes, because the children partition the alphabet interval of their parent. As a consequence, the *accumulated* size of any node is retained in its right child, as can be seen in Figure 4b. Since the $C$ array encodes the accumulated sizes of the nodes on level $h$, it also implicitly encodes the accumulated sizes of all nodes on levels $\ell < h$. Following this notion, we can conclude the following relations:

$$v(\ell, i) = \left\lfloor \frac{\min\{x \mid C[x] > i\} - 1}{2^{h-\ell}} \right\rfloor$$

and

$$p(\ell, v) = C[v \cdot 2^{h-\ell}]. \tag{1}$$

If the $C$ array is stored in ascending order, the minimum query required to find $v$ can be answered in time $\mathcal{O}(\log \sigma)$ using binary search. However, we seek a computation in constant time. We construct a bit vector $B_C$ of length $n$ and set $B_C[k] \coloneqq 1$ if $C[c] = k - 1$ for some $c$ and $B_C[k] \coloneqq 0$ otherwise and prepare it for constant-time rank queries. This can be done in time $\mathcal{O}(n)$ and requires $n + o(n)$ bits of additional space. $B_C$ marks the node boundaries on level $h$ of the wavelet tree, see Figure 4a for an example. We can now compute

$$v(\ell, i) = \left\lfloor \frac{\mathrm{rank}_1(B_C, i) - 1}{2^{h-\ell}} \right\rfloor \tag{2}$$

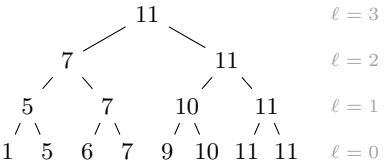
in constant time.

We now know that the $i$-th bit in $B_\ell^T$ corresponds to the $(\delta_v)$-th bit in the $v$-th node on level $\ell$ in the wavelet tree. We can compute $v$, $p$ and $\delta_v$ in constant time using the $C$ array and rank-enhanced bit vector $B_C$, which together occupy $\sigma\lceil \log n \rceil + n(1 + o(1))$ bits of space. Asymptotically, this space boundary matches that of the data structure presented by Fischer et al. [4].

*Example 1.* Figure 4, in combination with Figure 3, shows an example of the data structure for $T =$ `wavelettree`. Assume that we are interested in locating the node for bit $i = 9$ on level $\ell = 2$. With Equation 2, we get $v(2, 9) = \left\lfloor \frac{\mathrm{rank}_1(B_C, 9) - 1}{2^{3-2}} \right\rfloor = \left\lfloor \frac{5}{2} \right\rfloor = 2$.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\ell = 0$: | w | a | v | e | l | e | t | t | r | e | e |
| $\ell = 1$: | a | e | l | e | r | e | e | w | v | t | t |
| $\ell = 2$: | a | e | e | e | e | l | r | v | t | t | w |
| $\ell = 3$: | a | e | e | e | e | l | r | t | t | v | w |
| $B_C$: | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |

(a) The text re-ordering on each level and the bit vector $B_C$. The vertical lines mark the boundaries of the wavelet tree's nodes.

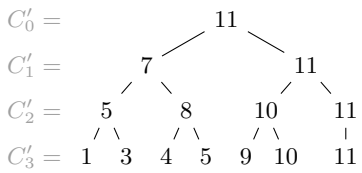



(b) The accumulated sizes of each of the wavelet tree's nodes. Note that the rightmost node on the bottom level corresponds to our artificial symbol $\top$ from Figure 3.

Figure 4: Display of the wavelet tree's text re-ordering on each level, including the virtual level $h = 3$, the bit vector $B_C$ and the accumulated node sizes for our running example text $T =$ `wavelettree`.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\ell = 0$: | w | a | v | e | l | e | t | t | r | e | e |
| $\ell = 1$: | a | e | l | e | r | e | e | w | v | t | t |
| $\ell = 2$: | a | e | e | e | e | v | t | t | l | r | w |
| $\ell = 3$: | a | t | t | l | w | e | e | e | e | v | r |

(a) The text re-ordering on each level of the wavelet matrix. The vertical lines mark the boundaries of the *nodes* of the wavelet matrix.




(b) The accumulated sizes of each of the *nodes* of the wavelet matrix. Note that $C'_3$, the bottom level, is not actually needed and depicted only for the sake of completeness.

Figure 5: Display of the wavelet matrix's text re-ordering on each level for running example text $T =$ `wavelettree`.

This means that the bit belongs to the third node on level 2 (because we start counting at zero). Furthermore, with Equation 1, we get $p(2, 2) = C[2 \cdot 2^{3-2}] = C[4] = 7$. This means that the third node on level 2 starts at position 7. Finally, it is $\delta_v(2, 9) = 9 - p(2, 2) = 9 - 7 = 2$, so bit 9 on level 2 ultimately corresponds to the third bit of the third node on that level.

**Locating in the wavelet matrix.** The question is how a similar locating can be done for the wavelet matrix. As previously mentioned, the bit vector $B_\ell^M$ of the wavelet matrix is the concatenation of the wavelet tree's node bit vectors on level $\ell$ in bit-reverse order.

We consider the situation where a wavelet matrix constructor sets the $j$-th bit of bit vector $B_\ell^M$ and are interested in the node to which this bit belongs. Analogously to $v$, $p$ and $\delta_v$, we define $u(\ell, j)$, $q(\ell, u)$ and $\delta_u(\ell, j) \coloneqq j - q(\ell, u(\ell, i))$ as the node into which the written bit belongs, the position of the node's first bit in $B_\ell^M$ and the distance of $j$ from the node's first bit, respectively.

Due to the re-ordering of the nodes, the correspondences between their accumulated sizes and the $C$ array, which we observed for the wavelet tree, are no longer valid for the wavelet matrix. As a consequence, we need to find a different way to compute $u$ and $q$.

The following observation is useful to find $u$: in both the wavelet tree and the wavelet matrix [2, Prop. 1], all occurrences of a symbol $c \in \Sigma'$ belong to the same node on any level. Therefore, in order to find the node to which any occurrence of $c$ belongs on virtual level $h$, it suffices to know to which node the *first* occurrence of $c$ belongs. This first occurrence of $c$ on level $h$ is always located at position $C[c]$. As seen previously, once the node for level $h$ is known, it is easy to narrow it down to any level $\ell < h$. Of course, we then have the node in the wavelet *tree*, but in the wavelet matrix, the nodes are simply permuted in bit-reverse order. Let $c$ be the symbol from which we computed the bit that we are setting in $B_\ell^M$. If $c$ is known, we can express

$$u(\ell, j, c) = \text{bitrev}_\ell(v(\ell, C[c])). \tag{3}$$

The consequences of having to know $c$ are discussed later.

It remains to compute $q$. As stated above, the $C$ array cannot be used directly to compute the accumulated node sizes for the wavelet matrix, because nodes are permuted. However, the node sizes themselves remain the same and thus, with awareness

of the bit-reversal ordering of nodes on every level, it is easy to precompute the accumulated node sizes for all nodes of the wavelet matrix using the $C$ array in time $\mathcal{O}(\sigma)$. Since we are dealing with a full binary tree of height $h = \log \sigma$, the accumulated wavelet matrix node sizes can be stored in an array $C'$ of length $2^h - 1 = \sigma - 1$ (since $\sigma$ is a power of two), occupying $(\sigma - 1)\lceil \log n \rceil$ bits of space. Figure 5b shows an example. We imagine $C'$ to be a set of arrays $C'_\ell$ for each level $\ell$, so that the first entry of $C'_\ell$ contains the size of the first node on level $\ell$. Then, $q$ can be found as follows:

$$q(\ell, u) = \begin{cases} 0 & \text{if } u = 0. \\ C'_\ell[u-1] & \text{if } u > 0. \end{cases} \tag{4}$$

We then know that the $j$-th bit in $B^M_\ell$ of the wavelet matrix corresponds to the $\delta_u$-th bit in the $u$-th node's bit vector on level $\ell$. We can compute $u$, $q$ and $\delta_u$ in constant time using the arrays $C$ and $C'$ and rank-enhanced bit vector $B_C$, which, in total, occupy $(2\sigma - 1)\lceil \log n \rceil + n(1 + o(1))$ bits of space.

*Example 2.* Figure 5, in combination with Figure 4 and Figure 3, shows an example for the data structure for $T = \texttt{wavelettree}$. Assume that we are interested in locating the node for bit $j = 9$ on level $\ell = 2$ of the wavelet matrix. The symbol for which the bit is written is $c = \texttt{r}$ (see Figure 5a). With Equation 3, we get $u(2, 9, \texttt{r}) = \text{bitrev}_3(v(2, C[\texttt{r}])) = \text{bitrev}_3(v(2, 6)) = \text{bitrev}_2(1) = 2$. This means that the bit belongs to the third node on level 2. Furthermore, with Equation 4, we get $q(2, 2) = C'_2[2 - 1] = 8$. This means that the third node on level 2 starts at position 8. Finally, it is $\delta_u(2, 9) = 9 - 8 = 1$, so bit 9 on level 2 ultimately corresponds to the second bit of the third node on that level.

### 3.2 Translating Between Wavelet Tree and Wavelet Matrix Construction

Using the locating data structures described above, we can express functions $f$ and $f^{-1}$ as follows:

$$f(\ell, i) = q(\ell, \text{bitrev}_\ell(v(\ell, i))) + \delta_v(\ell, i),$$
$$f^{-1}(\ell, j, c) = p(\ell, \text{bitrev}_\ell(u(\ell, j, c))) + \delta_u(\ell, j, c).$$

Both $f$ and $f^{-1}$ can be computed in constant time using the arrays $C$, $C'$ and rank-enhanced bit vector $B_C$. These occupy $\sigma\lceil \log n \rceil + (2\sigma - 1)\lceil \log n \rceil + n(1 + o(1))$ bits of space can be constructed in time $\mathcal{O}(\sigma + n)$.

**Limitations.** We impose the restriction that for $f^{-1}$, the symbol $c$, for which a bit is being set in $B^M_\ell$, has to be known when setting the bit. Even though this bit must ultimately have been computed from $c$, there are construction algorithms for the wavelet tree that redistribute the bits of $c$ before constructing the bit vectors [1, 7, 9, 11]. Due to the existence of our function $f$ alone, such techniques may as well be used for the construction of the wavelet matrix. In this case, $c$ is *not* known when setting the bit in question and $f^{-1}$ cannot be used.

More generally, in the wavelet tree, $c$ is always implicitly given by the tree structure itself and implicitly used by $f$ by jumping to the virtual bottom level to the leaf that would represent $c$ via the $C$ array. The wavelet matrix discards the tree structure and the information is lost, so that we need to receive it from the constructor in order to compute $f^{-1}$.

## 4 Conclusions

We solved an open theoretical problem concerning the construction of wavelet trees and wavelet matrices. We described a data structure that can be used to extend a construction algorithm for the wavelet matrix to construct instead the wavelet tree with constant time overhead. This data structure can be constructed in time $\mathcal{O}(\sigma+n)$ time and it requires $\mathcal{O}(\sigma \log n+n)$ bits of memory, matching the asymptotic time and space requirements of the data structure described by Fischer et al. [4] for the inverse direction, transforming wavelet tree construction into wavelet matrix construction.

However, because the wavelet matrix discards the wavelet tree's binary tree structure, we require some additional information from the constructor for our computations. This limitation makes our data structure unsuitable for the class of wavelet matrix constructors that do not keep the entire binary representation of the input symbols when computing the bit vectors. To that end, it is still open whether there is a data structure for our translation function with the same (or lower) asymptotic time and space requirements that does not require any information other than the position of the written bit in the wavelet matrix.

## Acknowledgements

We would like to thank Johannes Fischer and Florian Kurpicz from the TU Dortmund University's Chair of Algorithm Engineering for the motivation of this work and the supportive discussions related to the topic.

## References

1. M. A. Babenko, P. Gawrychowski, T. Kociumaka, and T. A. Starikovskaya: *Wavelet trees meet suffix trees*, in 26th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), SIAM, 2015, pp. 572–591.
2. F. Claude, G. Navarro, and A. O. Pereira: *The wavelet matrix: An efficient wavelet tree for large alphabets.* Inf. Syst., 47 2015, pp. 15–32.
3. P. Ferragina, R. Giancarlo, and G. Manzini: *The myriad virtues of wavelet trees.* Inform. and Comput., 207(8) 2009, pp. 849–866.
4. J. Fischer, F. Kurpicz, and M. Löbel: *Simple, fast and lightweight parallel wavelet tree construction*, in 20th Workshop on Algorithm Engineering and Experiments (ALENEX), SIAM, 2018, pp. 9–20.
5. R. Grossi, A. Gupta, and J. S. Vitter: *High-order entropy-compressed text indexes*, in 14th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), SIAM, 2003, pp. 841–850.
6. G. Jacobson: *Space-efficient static trees and graphs*, in 30th Symposium on Foundations of Computer Science (FOCS), IEEE, 1989, pp. 549–554.
7. Y. Kaneta: *Fast wavelet tree construction in practice*, in 25th International Symposium on String Processing and Information Retrieval (SPIRE), Springer, 2018, pp. 218–232.
8. V. Mäkinen and G. Navarro: *Position-restricted substring searching*, in 7th Latin American Theoretical Informatics Symposium (LATIN), vol. 3887 of Lecture Notes in Computer Science, Springer, 2006, pp. 703–714.
9. J. I. Munro, Y. Nekrich, and J. S. Vitter: *Fast construction of wavelet trees.* Theor. Comput. Sci., 638 2016, pp. 91–97.
10. G. Navarro: *Wavelet trees for all.* J. Discrete Algorithms, 25 2014, pp. 2–20.
11. G. Tischler: *On wavelet tree construction*, in 22nd Annual Symposium on Combinatorial Pattern Matching (CPM), Springer, 2011, pp. 208–218.