

Quasi-linear Time Computation of the Abelian Periods of a Word

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Abstract. In the last couple of years many research papers have been devoted to Abelian complexity of words. Recently, Constantinescu and Ilie (Bulletin EATCS 89, 167–170, 2006) introduced the notion of *Abelian period*. In this article we present two quadratic brute force algorithms for computing Abelian periods for special cases and a quasi-linear algorithm for computing all the Abelian periods of a word.

Keywords: Abelian period, Abelian repetition, weak repetition, design of algorithms, text algorithms, combinatorics on words

1 Introduction

An integer $p > 0$ is a (classical) period of a word w of length n if $w[i] = w[i + p]$ for any $1 \leq i \leq n - p$. Classical periods have been extensively studied in combinatorics on words [16] due to their direct applications in data compression and pattern matching.

The Parikh vector of a word w enumerates the cardinality of each letter of the alphabet in w . For example, given the alphabet $\Sigma = \{a, b, c\}$, the Parikh vector of the word $w = aaba$ is $(3, 1, 0)$. The reader can refer to [6] for a list of applications of Parikh vectors.

An integer p is an *Abelian period* for a word w over a finite alphabet $\Sigma = \{a_1, a_2, \dots, a_\sigma\}$ if w can be written as $w = \mathbf{u}_0 \mathbf{u}_1 \cdots \mathbf{u}_{k-1} \mathbf{u}_k$ where for $0 < i < k$ all the \mathbf{u}_i 's have the same Parikh vector \mathcal{P} such that $\sum_{i=1}^{\sigma} \mathcal{P}[i] = p$ and the Parikh vectors of \mathbf{u}_0 and \mathbf{u}_k are contained in \mathcal{P} [11]. For example, the word $w = ababbbabb$ can be written as $w = \mathbf{u}_0 \mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3$, with $\mathbf{u}_0 = a$, $\mathbf{u}_1 = bab$, $\mathbf{u}_2 = bba$ and $\mathbf{u}_3 = bb$, and 3 is an Abelian period of w with Parikh vector $(1, 2)$ over $\Sigma = \{a, b\}$.

This definition of Abelian period matches that of *weak repetition* (also called *Abelian power*) when \mathbf{u}_0 and \mathbf{u}_k are the empty word and $k > 2$ [12].

In the last couple of years many research papers have been devoted to Abelian complexity [13,1,8,3,14,2,4,20]. Efficient algorithms for Abelian Pattern Matching (also known as Jumbled Pattern Matching) have been designed [10,5,6,17,18,7].

Recently [15] gave algorithms for computing all the Abelian periods of a word of length n in time $O(n^2 \times \sigma)$. This was improved to time $O(n^2)$ in [9].

In this article we present a quasi-linear time algorithm for computing the Abelian periods of a word. In Section 2 we give some basic definitions and notation. Section 3 presents brute force algorithms while Section 4 presents our main contribution. Finally, Section 5 contains conclusions and perspectives.

2 Notation

Let $\Sigma = \{a_1, a_2, \dots, a_\sigma\}$ be a finite ordered alphabet of cardinality σ and Σ^* the set of words on alphabet Σ . We denote by $|\mathbf{w}|$ the length of the word \mathbf{w} . We write $\mathbf{w}[i]$ for the i -th symbol of \mathbf{w} and $\mathbf{w}[i..j]$ for the factor of \mathbf{w} from the i -th symbol to the j -th symbol, with $1 \leq i \leq j \leq |\mathbf{w}|$. We denote by $|\mathbf{w}|_a$ the number of occurrences of the symbol $a \in \Sigma$ in the word \mathbf{w} .

The *Parikh vector* of a word \mathbf{w} , denoted by $\mathcal{P}\mathbf{w}$, counts the occurrences of each letter of Σ in \mathbf{w} ; that is $\mathcal{P}\mathbf{w} = (|\mathbf{w}|_{a_1}, \dots, |\mathbf{w}|_{a_\sigma})$. Notice that two words have the same Parikh vector if and only if one word is a permutation of the other.

Given the Parikh vector $\mathcal{P}\mathbf{w}$ of a word \mathbf{w} , we denote by $\mathcal{P}\mathbf{w}[i]$ its i -th component and by $|\mathcal{P}\mathbf{w}|$ the sum of its components. Thus for $\mathbf{w} \in \Sigma^*$ and $1 \leq i \leq \sigma$, we have $\mathcal{P}\mathbf{w}[i] = |\mathbf{w}|_{a_i}$ and $|\mathcal{P}\mathbf{w}| = \sum_{i=1}^{\sigma} \mathcal{P}\mathbf{w}[i] = |\mathbf{w}|$.

Finally, given two Parikh vectors \mathcal{P}, \mathcal{Q} , we write $\mathcal{P} \subset \mathcal{Q}$ if $\mathcal{P}[i] \leq \mathcal{Q}[i]$ for every $1 \leq i \leq \sigma$ and $|\mathcal{P}| < |\mathcal{Q}|$.

Definition 1 ([11]). A word \mathbf{w} has an Abelian period (h, p) if $\mathbf{w} = \mathbf{u}_0 \mathbf{u}_1 \cdots \mathbf{u}_{k-1} \mathbf{u}_k$ such that:

- $\mathcal{P}\mathbf{u}_0 \subset \mathcal{P}\mathbf{u}_1 = \cdots = \mathcal{P}\mathbf{u}_{k-1} \supset \mathcal{P}\mathbf{u}_k$,
- $|\mathcal{P}\mathbf{u}_0| = h$, $|\mathcal{P}\mathbf{u}_1| = p$.

We call \mathbf{u}_0 and \mathbf{u}_k resp. the *head* and the *tail* of the Abelian period. Notice that the length $t = |\mathbf{u}_k|$ of the tail is uniquely determined by h, p and $|\mathbf{w}|$, namely $t = (|\mathbf{w}| - h) \bmod p$.

The following lemma gives a bound on the maximum number of Abelian periods of a word.

Lemma 2 ([15]). The maximum number of Abelian periods for a word of length n over the alphabet Σ is $\Theta(n^2)$.

Proof. The word $(a_1 a_2 \cdots a_\sigma)^{n/\sigma}$ has Abelian period (h, p) for any $p \equiv 0 \pmod{\sigma}$ and $h < p$. □

A natural order can be defined on the Abelian periods.

Definition 3. Two distinct Abelian periods (h, p) and (h', p') of a word \mathbf{w} are ordered as follows: $(h, p) < (h', p')$ if $p < p'$ or $(p = p'$ and $h < h')$.

Definition 4 ([9]). Let \mathbf{w} be a word of length n . Then the mapping $pr : \Sigma \rightarrow A$, where A is the set of the first σ prime numbers, is defined by:

$$pr(\sigma_i) = i\text{-th prime number.}$$

The P -signature of \mathbf{w} is defined by:

$$P\text{-signature}(\mathbf{w}) = \prod_{i=1}^n pr(\mathbf{w}[i]).$$

Definition 5 ([9]). Let \mathbf{w} be a word of length n . Then the mapping $s : \Sigma \rightarrow B$, where B is the set of the first $\sigma - 1$ powers of $n + 1$ and 0, is defined by:

$$s(\sigma_i) = \begin{cases} 0 & \text{if } i = 1 \\ (n + 1)^{i-2} & \text{otherwise.} \end{cases}$$

The S -signature of \mathbf{w} is defined by:

$$S\text{-signature}(\mathbf{w}) = \sum_{i=0}^n s(\mathbf{w}[i]).$$

Observation 1 ([9]) For a word \mathbf{w} of length n the array Pr of n elements is defined by

$$Pr[i] = \prod_{j=1}^i pr(\mathbf{w}[j]),$$

then

$$P\text{-signature}(\mathbf{w}[k.. \ell]) = \begin{cases} Pr[\ell]/Pr[k-1] & \text{if } k \neq 0 \\ Pr[\ell] & \text{otherwise.} \end{cases}$$

Observation 2 ([9]) For a word \mathbf{w} of length n the array S of n elements is defined by

$$S[i] = \sum_{j=1}^i s(\mathbf{w}[j]),$$

then

$$S\text{-signature}(\mathbf{w}[k.. \ell]) = \begin{cases} S[\ell] - S[k-1] & \text{if } k \neq 0 \\ S[\ell] & \text{otherwise.} \end{cases}$$

Example 6. $\mathbf{w} = \text{abaab}$:

i	1	2	3	4	5
$\mathbf{w}[i]$	a	b	a	a	b
$pr(\mathbf{w}[i])$	2	3	2	2	3
$Pr[i]$	2	6	12	24	72

$$\begin{aligned} P\text{-signature}(\mathbf{w}[3..5]) &= \\ P\text{-signature}(\text{aab}) &= \\ Pr[5]/Pr[2] &= 72/6 = 12 \end{aligned}$$

i	1	2	3	4	5
$\mathbf{w}[i]$	a	b	a	a	b
$s(i)$	0	1	0	0	1
$S[i]$	0	1	1	1	2

$$\begin{aligned} S\text{-signature}(\mathbf{w}[3..5]) &= \\ S\text{-signature}(\text{aab}) &= \\ S[5] - S[2] &= 2 - 1 = 1 \end{aligned}$$

3 Brute Force Algorithms

We will first focus on the case where we consider periods without head nor tail.

In the remaining of the article we will write that a word \mathbf{w} has Abelian period p whenever it has Abelian period $(0, p)$. When the tail is also empty, for a word \mathbf{w} of length n an Abelian period p must divide n . We define:

- $P[i]$ is the set of Abelian periods of $\mathbf{w}[1..i]$;
- $V[i] = \mathcal{P}(\mathbf{w}[1..i])$ is the Parikh vector of $\mathbf{w}[1..i]$.

3.1 Abelian periods with neither head nor tail

In a first step we set $P[i] = \{i\}$ for all the divisors of n . Then we process the positions i of \mathbf{w} in ascending order: if $j \in P[i]$ and $\mathcal{P}_{\mathbf{w}}[i+1..i+j] = \mathcal{P}_{\mathbf{w}}[1..j]$, then we add j to $P[i+j]$. This test can be done in $O(\sigma)$ time by precomputing the Parikh vectors of all the prefixes of \mathbf{w} or in constant time using signatures. At the end of the process $P[n]$ contains all the Abelian periods of \mathbf{w} with neither head nor tail (see algorithm in Figure 1).

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ABELIANPERIODSNOHEADNOTAIL( $w, n$ )
1  $V[i] \leftarrow \mathcal{P}(w[1..i]), \forall 1 \leq i \leq n$ 
2  $P[i] \leftarrow \emptyset, \forall 1 \leq i \leq n$ 
3 for  $i \leftarrow 1$  to  $n/2$  do
4   if  $n \bmod i = 0$  then
5      $P[i] \leftarrow \{i\}$ 
6 for  $i \leftarrow 1$  to  $n-1$  do
7   for  $j \in P[i]$  do
8     if  $V[i+j] - V[i] = V[j]$  then
9        $P[i+j] \leftarrow P[i+j] \cup \{j\}$ 
10 return  $P[n]$ 

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Figure 1. Compute the Abelian periods with no head and no tail of a word w of length n

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ABELIANPERIODSNOHEADWITHTAIL( $w, n$ )
1  $V[i] \leftarrow \mathcal{P}(w[1..i]), \forall 1 \leq i \leq n$ 
2  $P[i] \leftarrow \{i\}, \forall 1 \leq i \leq n/2$ 
3  $P[i] \leftarrow \emptyset, \forall n/2 < i \leq n$ 
4 for  $i \leftarrow 1$  to  $n-1$  do
5   for  $j \in P[i]$  do
6     if  $i+j > n$  then
7       if  $V[n] - V[i+1] \leq V[j]$  then
8          $P[n] \leftarrow P[n] \cup \{j\}$ 
9       else if  $V[i+j] - V[i] = V[j]$  then
10         $P[i+j] \leftarrow P[i+j] \cup \{j\}$ 
11 return  $P[n]$ 

```

Figure 2. Compute the Abelian periods without head and with a possibly non-empty tail of a word w of length n

Example 7. $w = \text{abaababbbabaabbabbaaabbababbaa}$:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$w[i]$	a	b	a	a	b	a	b	b	b	a	b	a	a	b	b	a	b	b	a	a	a	b	b	a	b	a	b	b	a	a
P	{1}	{2}	{3}		{5}	{6}				{10}					{15}						{10}									{10}
						{3}																								

Theorem 8. *The algorithm **AbelianPeriodsNoHeadNoTail** computes all the Abelian periods with neither head nor tail of a word w of length n in time $O(n^2 \times \sigma)$ if the test in line 8 is performed by comparing Parikh vectors and in time $O(n^2)$ if the test in line 8 is performed by using S -signatures or P -signatures.*

3.2 Abelian periods without head with tail

Now we consider Abelian periods without head and with a possibly non-empty tail. We adapt the previous algorithm by setting $P[i] = \{i\}$ for $1 \leq i \leq n/2$ (see algorithm Figure 2).

Theorem 9. *The algorithm **AbelianPeriodsNoHeadWithTail** computes all the Abelian periods without head and with tail of a word w of length n in time $O(n^2 \times \sigma)$ if the tests in lines 7 and 1 are performed by comparing Parikh vectors and in time $O(n^2)$ if the test in lines 7 and 1 are performed by using S -signatures or P -signatures.*

4 Quasi-Linear Time Computation of Abelian Periods with neither Head nor Tail

In a linear-time preprocessing phase we compute $\mathcal{P}_{\mathbf{w}}[j]$, $j = 1, 2, \dots, \sigma$, the components of the Parikh vector of the word \mathbf{w} . Also we compute

$$g = \gcd(\mathcal{P}_{\mathbf{w}}[1], \mathcal{P}_{\mathbf{w}}[2], \dots, \mathcal{P}_{\mathbf{w}}[\sigma])$$

and $q = n/g$. Without loss of generality we suppose $\sigma \geq 2$ and $g > 1$. In $O(\sqrt{g})$ time we compute a stack D of all divisors $1 \leq d \leq g$ of g in ascending order.

Definition 10. *The word \mathbf{w} is an **Abelian repetition** of period p and exponent e if $p \mid n$ and each of the e substrings*

$$\mathbf{w}[1..p], \mathbf{w}[p+1..2p], \dots, \mathbf{w}[n-p+1..n]$$

contains $(p \times \mathcal{P}_{\mathbf{w}}[j])/n = \mathcal{P}_{\mathbf{w}}[j]/e$ occurrences of the letter $\sigma_j \in \Sigma$ for any j .

In other words, an Abelian repetition of period p and exponent e is the concatenation of e strings all having the same Parikh vector \mathcal{P} of length p .

Observation 3 *The only possible Abelian periods p of \mathbf{w} are of the form $p = d \times q$, where d is an entry in D . Thus the smallest period is $d \times q$, where d is the least such entry. (Note that the last element of D is g .)*

Definition 11 (Segment). *A factor $\mathbf{w}[i..j]$ is a **segment** of \mathbf{w} if:*

1. $i = k \times q + 1$ with $k \geq 0$;
2. $j - i + 1 = t \times q$ with $t \geq 1$;
3. $\mathcal{P}_{\mathbf{w}[i..j][k]}/(j - i + 1) = \mathcal{P}_{\mathbf{w}}[k]/|\mathbf{w}|$ for every letter $\sigma_k \in \Sigma$;
4. there does not exist a $j' < j$ such that $j' - i + 1 = t' \times q$ and $\mathcal{P}_{\mathbf{w}[i..j'][k]}/(j' - i + 1) = \mathcal{P}_{\mathbf{w}}[k]/|\mathbf{w}|$ for every letter $\sigma_k \in \Sigma$.

In other words segments:

- start at positions multiples of q plus one;
- are non-empty and of length multiple of q ;
- have the same proportion of every letter as the whole word \mathbf{w} ;
- are of minimal length.

Since we suppose that \mathbf{w} has Abelian period $p \in 1..n/2$, it follows that either \mathbf{w} itself is a segment or else consists of a concatenation of segments. Note that a segment is a minimum-length substring of Abelian period p .

Lemma 12. *The word \mathbf{w} has Abelian period $d \times q$ if and only if for every $k = 0, 1, \dots, n/(d \times q) - 1$, $k \times d \times q + 1$ is the starting position of a segment of \mathbf{w} .*

We begin by computing the segments of \mathbf{w} (see Figure 3), making use of the precomputed values q and $\mathcal{P}_{\mathbf{w}}$. We compute a Boolean array L of n elements: for $1 \leq i \leq n$, $L[i] = 1$ iff i is the starting position of a segment, $L[i] = 0$ otherwise.

Observation 4 *If p is an Abelian period of \mathbf{w} with neither head nor tail and T is the length of the longest segment of \mathbf{w} divided by q , then $p \geq T$.*

```

COMPUTESSEGMENTS( $\mathbf{w}, n, q, \mathcal{P}\mathbf{w}$ )
1  ( $i, T$ )  $\leftarrow$  (1, 0)
2   $L \leftarrow 0^n$ 
3  while  $i \leq n$  do
     $\triangleright$  Start a new segment
4  ( $i_0, j, t, \text{count}$ )  $\leftarrow$  ( $i, 0, 0, 0^\sigma$ )
5  while  $j \leq \sigma$  do
     $\triangleright$  See if  $t$  partitions of length  $q$  form a segment
6   $t \leftarrow t + 1$ 
7  for  $k \leftarrow 1$  to  $q$  do
8   $j \leftarrow \mathbf{w}[i]$ 
9   $\text{count}[j] \leftarrow \text{count}[j] + 1$ 
10  $i \leftarrow i + 1$ 
     $\triangleright$  Check counts of letters  $1..j$  from position  $i_0$ 
11  $j \leftarrow 1$ 
12  $t' \leftarrow t \times q$ 
13 while  $j \leq \sigma$  and  $\text{count}[j] = (t' \times \mathcal{P}\mathbf{w}[j])/n$  do
14  $j \leftarrow j + 1$ 
     $\triangleright$  Update the array  $L$  and the maximum segment length  $T$ 
15  $L[i_0] \leftarrow 1$ 
16  $T \leftarrow \max\{T, t\}$ 
17 return ( $L, T$ )

```

Figure 3. Compute a Boolean array L of the starting positions of the segments of \mathbf{w} ordered from left to right, also the maximum number T of factors of length q in any segment

The procedure that computes L visits each position i in \mathbf{w} once, and corresponding to each i performs constant-time processing: the internal **while** loop updates j at most σ times corresponding to each partition of length $q \geq \sigma$.

Proposition 13. *The algorithm COMPUTESSEGMENTS($\mathbf{w}, n, q, \mathcal{P}\mathbf{w}$) computes the segments of a word \mathbf{w} of length n on an alphabet of size σ in time $O(n)$.*

Example 14. $\mathbf{w} = \text{abaababbbabaabbabbaaabbababbaa}$: $n = 30$, $\mathcal{P}\mathbf{w} = (15, 15)$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\mathbf{w}[i]$	a	b	a	a	b	a	b	b	b	a	b	a	a	b	b	a	b	b	a	a	a	b	b	a	b	a	b	b	a	a
$L[i]$	1	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	0	0
T	0	1	1	1	1	1	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3

\mathbf{w} is thus a concatenation of segments: $\mathbf{w} = \text{ab} \cdot \text{aababb} \cdot \text{ba} \cdot \text{ba} \cdot \text{ab} \cdot \text{ba} \cdot \text{bbaa} \cdot \text{ab} \cdot \text{ba} \cdot \text{ba} \cdot \text{bbaa}$ and $T = 3$.

The procedure, given in Figure 4, scans all the multiples of the divisors $d \in D$, their number is equal to the sum of the divisors of g which is in $O(n \log \log n)$ [19].

In practice, the case where $d = 1$ is treated in lines 5 and 7. If $T = 1$, it means that w can be segmented into factors of length q : q is then an Abelian period of w . The case where $d = g$ is treated outside the main loop, at the end of the algorithm: it corresponds to the trivial case where the Abelian period is n .

Example 15. $\mathbf{w} = \text{abaababbbabaabbabbaaabbababbaa}$: $n = 30$, $\mathcal{P}\mathbf{w}[1] = \mathcal{P}\mathbf{w}[2] = 15$, $g = 15$, $q = 2$, $D = (1, 3, 5, 15)$ and $T = 3$. Since $T \neq 1$, q is not an Abelian period: case $d = 1$ is done. When $d = 3$, $p = 7$ and 7 is not a starting position of a segment. When $d = 5$, $p = 11$ and 11 is a starting position of a segment then $p = 21$

```

COMPUTESPERIOD( $\mathbf{w}, n$ )
1  Compute  $\mathcal{P}_{\mathbf{w}, g, D}$ 
2   $q \leftarrow n/g$ 
3   $(L, T) \leftarrow \text{COMPUTESSEGMENTS}(\mathbf{w}, n, q, \mathcal{P}_{\mathbf{w}})$ 
4   $R \leftarrow \emptyset$ 
    $\triangleright$  Deal quickly with easy cases
5  if  $T = 1$  then
6     $R \leftarrow R \cup \{q\}$ 
7     $d \leftarrow \text{POP}(D)$ 
    $\triangleright$  Fast forward in  $D$  past impossible cases
8  repeat
9     $d \leftarrow \text{POP}(D)$ 
10 until  $d \geq T$ 
11 while  $d < g$  do
12    $p \leftarrow d \times q + 1$ 
    $\triangleright$  Test if all multiples of  $p$  are starting positions of segments
13   while  $p < n$  do
14     if  $L[p] = 1$  then
15        $p \leftarrow p + d \times q$ 
16     else break
17   if  $p \geq n$  then
18      $R \leftarrow R \cup \{d \times q\}$ 
19    $d \leftarrow \text{POP}(D)$ 
20 if  $q \neq n$  then
21    $R \leftarrow R \cup \{n\}$ 
22 return  $R$ 

```

Figure 4. In ascending order of divisors d of g , use the array L to determine whether or not \mathbf{w} is an Abelian repetition of period $d \times q$

and 21 is a starting position of a segment: 10 is an Abelian period. The case where $d = 15$ is trivial since it corresponds to Abelian period n . Thus the algorithm returns $\{10, 30\}$. In the worst case the algorithm could have scanned all the multiples of 3 (they are 5) and all the multiples of 5 (they are 3) less than or equal to 15.

Theorem 16. *The algorithm $\text{COMPUTESPERIOD}(\mathbf{w}, n)$ computes all the Abelian periods of \mathbf{w} in time $O(n \log \log n)$.*

5 Conclusions and perspectives

In this article we gave brute force algorithms for computing Abelian periods for a word \mathbf{w} of length n in the two following cases: no head, no tail and no head with tail. These algorithms run in time $O(n^2)$ but is this complexity tight? We also present a quasi-linear time algorithm for computing all the Abelian periods of a word in the case no head, no tail. Does an algorithm of the same complexity exist for a word \mathbf{w} of length at most $n + q - 1$ containing a substring of length n that is an Abelian repetition with neither head nor tail of some period $dq \leq n$?

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